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A class of digit extraction BBP-type formulas in general binary bases

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Abstract: BBP-type formulas are usually discovered experimentally, one at a time and in specific bases, through computer searches. In this paper, however, we derive explicit digit extraction BBP-type formulas in general binary bases $b = 2^{12p}$, for $p \in \mathbb{Z}^+$ and $\mod (p, 2) = 1$. As particular examples, new binary formulas are presented for $\pi\sqrt{3}$, $\pi\sqrt{3}\log 2$, $\sqrt{3}\ \mathrm{Cl}_2(\pi/3)$ and a couple of other polylogaritm constants. A variant of the formula for $\pi\sqrt{3}\log 2$ derived in this paper has been known for over ten years but was hitherto unproved. Binary BBP-type formulas for the logarithms of an infinite set of primes and binary BBP-type representations for the arctangents of an infinite set of rational numbers are also presented. Finally, new binary BBP-type zero relations are established.
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1 Introduction

BBP-type formulas are formulas of the form

$$\alpha = \sum_{k=0}^{\infty} 1/b^k \sum_{j=1}^{l} a_j/(kl + j)^s$$

where s, b, l (degree, base, length respectively) and a_j are integers, and α is some constant. Formulas of this type were first introduced in a 1996 paper [1], where a formula of this type for π was given. Such formulas allow digit extraction — the i-th digit of a mathematical constant α in base b can be calculated directly, without needing to compute any of the previous $i - 1$ digits, by means of simple algorithms that do not require multiple-precision arithmetic [2].

Apart from digit extraction, another reason the study of BBP-type formulas has continued to attract attention is that BBP-type constants are conjectured to be either rational or normal to base b [3, 4, 5], that is their base-b digits are randomly distributed.

BBP-type formulas are usually discovered experimentally, one at a time and in specific bases, through computer searches. In this paper, however, we derive explicit digit extraction BBP-type formulas in general binary bases $b=2^{12p}$, for $p\in\mathbb{Z}^+$ and $\mod(p,2)=1$.

2 Definitions and notations

The polylogarithm functions denoted by Li in this paper are defined by

$$\mathrm{Li}_s[z]=\sum_{k=1}^{\infty}\frac{z^k}{k^s},\quad |z|\le 1\,, s\in\mathbb{Z}^+\,.$$

For $|z|=1$ and $x\in[0,2\pi]$ we have

$$\begin{aligned}\mathrm{Li}_{2n}[e^{ix}]&=\mathrm{Gl}_{2n}(x)+i\mathrm{Cl}_{2n}(x)\\ \mathrm{Li}_{2n+1}[e^{ix}]&=\mathrm{Cl}_{2n+1}(x)+i\mathrm{Gl}_{2n+1}(x)\,,\end{aligned}\tag{1}$$

where Gl and Cl are Clausen sums [6] defined, for $n\in\mathbb{Z}^+$ by

$$\begin{aligned}\mathrm{Cl}_{2n}(x)&=\sum_{k=1}^{\infty}\frac{\sin kx}{k^{2n}},\quad \mathrm{Cl}_{2n+1}(x)=\sum_{k=1}^{\infty}\frac{\cos kx}{k^{2n+1}}\\ \mathrm{Gl}_{2n}(x)&=\sum_{k=1}^{\infty}\frac{\cos kx}{k^{2n}},\quad \mathrm{Gl}_{2n+1}(x)=\sum_{k=1}^{\infty}\frac{\sin kx}{k^{2n+1}}\,.\end{aligned}\tag{2}$$

We shall find the following formulas useful:

$$\begin{aligned}\mathrm{Gl}_{2n}(x)&=(-1)^{1+[n/2]}2^{n-1}\pi^n\mathrm{B}_n(x/2\pi)/n!\\ \frac{1}{m^{n-1}}\mathrm{Cl}_n(mx)&=\sum_{r=0}^{m-1}\mathrm{Cl}_n(x+2\pi r/m)\,.\end{aligned}\tag{3}$$

Here $[n/2]$ denotes the integer part of $n/2$ and B_n are the Bernoulli polynomials defined by

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t-1}=\sum_{n=0}^{\infty}\frac{\mathrm{B}_n(x)t^n}{n!}\,.$$

In order to save space, we will give the BBP-type formulas using the compact P-notation [2]:

$$P(s,b,l,A)\equiv\sum_{k=0}^{\infty}\frac{1}{b^k}\sum_{j=1}^{l}\frac{a_j}{(kl+j)^s}\,,\tag{4}$$

where s, b and l are integers, and $A=(a_1,a_2,\ldots,a_l)$ is a vector of integers.

3 Degree s formulas

Using the identities

$$\operatorname{Re}\operatorname{Li}_s\left[\frac{1}{\sqrt{2}^p}\exp ix\right]=\sum_{k=1}^{\infty}\left[\frac{1}{\sqrt{2}^{pk}}\frac{1}{k^s}\cos kx\right] \tag{5}$$

and

$$\operatorname{Im}\operatorname{Li}_s\left[\frac{1}{\sqrt{2}^p}\exp ix\right]=\sum_{k=1}^{\infty}\left[\frac{1}{\sqrt{2}^{pk}}\frac{1}{k^s}\sin kx\right] \tag{6}$$

for $x\in\{\pi/12,5\pi/12,7\pi/12,11\pi/12\}$ and the fact that

$$\cos\left(\frac{\pi}{12}\right)=\frac{\sqrt{3}+1}{2\sqrt{2}}=\sin\left(\frac{7\pi}{12}\right)=\sin\left(\frac{5\pi}{12}\right)=-\cos\left(\frac{11\pi}{12}\right)$$

and

$$\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{3}-1}{2\sqrt{2}}=\cos\left(\frac{5\pi}{12}\right)=\sin\left(\frac{11\pi}{12}\right)=-\cos\left(\frac{7\pi}{12}\right),$$

it is not difficult to obtain the following results, written in the P-notation (Eq. (4)):

$$\begin{aligned}&\operatorname{Re}\operatorname{Li}_s\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{i\pi}{12}\right)\right]+\operatorname{Re}\operatorname{Li}_s\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{7i\pi}{12}\right)\right]\\&=\frac{1}{2^{12p}}P(s,2^{12p},24,(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},0,2^{(\frac{1}{2}-\frac{p}{2}+11p)},2^{10p},\\&-2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+8p)},-2^{8p},-2^{(\frac{1}{2}-\frac{p}{2}+8p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+6p)},\\&-2^{1+6p},-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},0,-2^{(\frac{1}{2}-\frac{p}{2}+5p)},-2^{4p},2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0,\\&-2^{(-\frac{1}{2}+\frac{p}{2}+2p)},2^{2p},2^{(\frac{1}{2}-\frac{p}{2}+2p)},0,2^{(-\frac{1}{2}+\frac{p}{2})},2))\,,\end{aligned} \tag{7}$$

$$\begin{aligned}&\operatorname{Re}\operatorname{Li}_s\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{i\pi}{12}\right)\right]-\operatorname{Re}\operatorname{Li}_s\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{7i\pi}{12}\right)\right]\\&=\frac{\sqrt{3}}{2^{12p}}P(s,2^{12p},24,(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},2^{11p},0,0,\\&2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+8p)},0,0,-2^{7p},-2^{(-\frac{1}{2}+\frac{p}{2}+6p)},\\&0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},-2^{5p},0,0,-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0,\\&2^{(-\frac{1}{2}+\frac{p}{2}+2p)},0,0,2^{p},2^{(-\frac{1}{2}+\frac{p}{2})},0))\,,\end{aligned} \tag{8}$$

$$\operatorname{Im}\operatorname{Li}_s\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{i\pi}{12}\right)\right]+\operatorname{Im}\operatorname{Li}_s\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{7i\pi}{12}\right)\right]$$
$$=\frac{\sqrt{3}}{2^{12p}}P(s,2^{12p},24,(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},0,0,2^{10p},$$
$$2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+8p)},2^{8p},0,0,2^{(-\frac{1}{2}+\frac{p}{2}+6p)},$$
$$0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},0,0,-2^{4p},-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0,$$
$$-2^{(-\frac{1}{2}+\frac{p}{2}+2p)},-2^{2p},0,0,-2^{(-\frac{1}{2}+\frac{p}{2})},0))\,, \tag{9}$$

$$\operatorname{Im}\operatorname{Li}_s\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{i\pi}{12}\right)\right]-\operatorname{Im}\operatorname{Li}_s\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{7i\pi}{12}\right)\right]$$
$$=\frac{1}{2^{12p}}P(s,2^{12p},24,(-2^{(-\frac{1}{2}+\frac{p}{2}+11p)},2^{11p},2^{(\frac{1}{2}-\frac{p}{2}+11p)},0,$$
$$2^{(-\frac{1}{2}+\frac{p}{2}+9p)},2^{1+9p},2^{(-\frac{1}{2}+\frac{p}{2}+8p)},0,2^{(\frac{1}{2}-\frac{p}{2}+8p)},2^{7p},-2^{(-\frac{1}{2}+\frac{p}{2}+6p)},$$
$$0,2^{(-\frac{1}{2}+\frac{p}{2}+5p)},-2^{5p},-2^{(\frac{1}{2}-\frac{p}{2}+5p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},-2^{1+3p},$$
$$-2^{(-\frac{1}{2}+\frac{p}{2}+2p)},0,-2^{(\frac{1}{2}-\frac{p}{2}+2p)},-2^{p},2^{(-\frac{1}{2}+\frac{p}{2})},0))\,, \tag{10}$$

$$\operatorname{Re}\operatorname{Li}_{\mathrm{s}}\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{5i\pi}{12}\right)\right]+\operatorname{Re}\operatorname{Li}_{\mathrm{s}}\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{11i\pi}{12}\right)\right]$$
$$=\frac{1}{2^{12p}}P\left(s,2^{12p},24,\left(-2^{(-\frac{1}{2}+\frac{p}{2}+11p)},0,\ -2^{(\frac{1}{2}-\frac{p}{2}+11p)},2^{10p},\right.\right.$$
$$2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+8p)},-2^{8p},2^{(\frac{1}{2}-\frac{p}{2}+8p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+6p)},$$
$$-2^{1+6p},2^{(-\frac{1}{2}+\frac{p}{2}+5p)},0,2^{(\frac{1}{2}-\frac{p}{2}+5p)},-2^{4p},-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0,$$
$$\left.\left.2^{(-\frac{1}{2}+\frac{p}{2}+2p)},2^{2p},-2^{(\frac{1}{2}-\frac{p}{2}+2p)},0,-2^{(-\frac{1}{2}+\frac{p}{2})},2\right)\right)\,, \tag{11}$$

$$\operatorname{Re}\operatorname{Li}_{\mathrm{s}}\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{5i\pi}{12}\right)\right]-\operatorname{Re}\operatorname{Li}_{\mathrm{s}}\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{11i\pi}{12}\right)\right]$$
$$=\frac{\sqrt{3}}{2^{12p}}P\left(s,2^{12p},24,\left(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},-2^{11p},\ 0,0,\right.\right.$$
$$2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+8p)},0,0,2^{7p},-2^{(-\frac{1}{2}+\frac{p}{2}+6p)},$$
$$0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},2^{5p},0,0,-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0,$$
$$\left.\left.2^{(-\frac{1}{2}+\frac{p}{2}+2p)},0,0,-2^{p},2^{(-\frac{1}{2}+\frac{p}{2})},0\right)\right)\,, \tag{12}$$

$$
\begin{aligned}
&\operatorname{Im}\operatorname{Li_s}\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{5i\pi}{12}\right)\right]+\operatorname{Im}\operatorname{Li_s}\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{11i\pi}{12}\right)\right]\\
&=\frac{\sqrt{3}}{2^{12p}}P\left(s,2^{12p},24,\left(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},0,\ 0,-2^{10p},\right.\right.\\
&2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+8p)},-2^{8p},0,0,2^{(-\frac{1}{2}+\frac{p}{2}+6p)},\\
&0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},0,0,2^{4p},-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0,\\
&\left.\left.-2^{(-\frac{1}{2}+\frac{p}{2}+2p)},2^{2p},0,0,-2^{(-\frac{1}{2}+\frac{p}{2})},0\right)\right)
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
&\operatorname{Im}\operatorname{Li_s}\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{5i\pi}{12}\right)\right]-\operatorname{Im}\operatorname{Li_s}\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{11i\pi}{12}\right)\right]\\
&=\frac{1}{2^{12p}}P\left(s,2^{12p},24,\left(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},2^{11p},\ -2^{(\frac{1}{2}-\frac{p}{2}+11p)},0,\right.\right.\\
&-2^{(-\frac{1}{2}+\frac{p}{2}+9p)},2^{1+9p},-2^{(-\frac{1}{2}+\frac{p}{2}+8p)},0,-2^{(\frac{1}{2}-\frac{p}{2}+8p)},2^{7p},2^{(-\frac{1}{2}+\frac{p}{2}+6p)},\\
&0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},-2^{5p},2^{(\frac{1}{2}-\frac{p}{2}+5p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+3p)},-2^{1+3p},\\
&\left.\left.2^{(-\frac{1}{2}+\frac{p}{2}+2p)},0,2^{(\frac{1}{2}-\frac{p}{2}+2p)},-2^{p},-2^{(-\frac{1}{2}+\frac{p}{2})},0\right)\right)\ .
\end{aligned}
\tag{14}
$$

We note that although the above formulas are true for all $p > 0$, they are BBP-type only for $p \in \mathbb{Z}^+$ and $\mod (p, 2) = 1$.

4 Degree 1 formulas

When $s = 1$, the polylogarithms on the left hand side in each of the above formulas can be evaluated, using the identities

$$\arctan\left(\frac{q\sin x}{1-q\cos x}\right)=\operatorname{Im}\operatorname{Li}_1\left[q\exp(ix)\right]$$

and

$$-\frac{1}{2}\log\left(1-2q\cos x+q^2\right)=\operatorname{Re}\operatorname{Li}_1\left[q\exp(ix)\right]\ ,$$

and we have the following degree 1 binary BBP-type formulas

$$
\begin{aligned}
&-\frac{1}{2}\log(1-2^{(\frac{1}{2}-\frac{p}{2})}+2^{-p}-2^{(\frac{1}{2}+\frac{p}{2}-2p)}+2^{-2p}) \\
&\quad=\frac{1}{2^{12p}}P(1,2^{12p},24,(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},0,2^{(\frac{1}{2}-\frac{p}{2}+11p)},2^{10p}, \\
&\quad -2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+8p)},-2^{8p},-2^{(\frac{1}{2}-\frac{p}{2}+8p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+6p)}, \\
&\quad -2^{1+6p},-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},0,-2^{(\frac{1}{2}-\frac{p}{2}+5p)},-2^{4p},2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0, \\
&\quad -2^{(-\frac{1}{2}+\frac{p}{2}+2p)},2^{2p},2^{(\frac{1}{2}-\frac{p}{2}+2p)},0,2^{(-\frac{1}{2}+\frac{p}{2})},2))\,,
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&\frac{\sqrt{3}}{2}\log\left(\frac{(1+2^{(-\frac{1}{2}-\frac{p}{2})}\sqrt{3}-2^{(-\frac{1}{2}-\frac{p}{2})}+2^{-p})^2}{1-2^{(\frac{1}{2}-\frac{p}{2})}-2^{(\frac{1}{2}-\frac{3p}{2})}+2^{-2p}+2^{-p}}\right) \\
&=\frac{3}{2^{12p}}P(1,2^{12p},24,(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},2^{11p},0,0,2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+8p)},0,0,-2^{7p}, \\
&-2^{(-\frac{1}{2}+\frac{p}{2}+6p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},-2^{5p},0,0,-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+2p)},0,0,2^{p},, \\
&2^{(-\frac{1}{2}+\frac{p}{2})},0))\,,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\sqrt{3}\arctan\left[\left(\frac{1-2^{\frac{p+1}{2}}}{1+2^{\frac{p+1}{2}}-2^{p+1}}\right)\sqrt{3}\right] \\
&=\frac{3}{2^{12p}}P(1,2^{12p},24,(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},0,0,2^{10p},2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+8p)},2^{8p},0,0,2^{(-\frac{1}{2}+\frac{p}{2}+6p)}, \\
&0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},0,0,-2^{4p},-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+2p)},-2^{2p},0,0,-2^{(-\frac{1}{2}+\frac{p}{2})},0))\,,
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&-\arctan\left[\frac{2^{\frac{1-p}{2}}-1}{-2^{\frac{1+p}{2}}+1}\right] \\
&\quad=\frac{1}{2^{12p}}P(1,2^{12p},24,(-2^{(-\frac{1}{2}+\frac{p}{2}+11p)},2^{11p},2^{(\frac{1}{2}-\frac{p}{2}+11p)}, \\
&0,2^{(-\frac{1}{2}+\frac{p}{2}+9p)},2^{1+9p},2^{(-\frac{1}{2}+\frac{p}{2}+8p)},0,2^{(\frac{1}{2}-\frac{p}{2}+8p)},2^{7p},-2^{(-\frac{1}{2}+\frac{p}{2}+6p)}, \\
&0,2^{(-\frac{1}{2}+\frac{p}{2}+5p)},-2^{5p},-2^{(\frac{1}{2}-\frac{p}{2}+5p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},-2^{1+3p}, \\
&-2^{(-\frac{1}{2}+\frac{p}{2}+2p)},0,-2^{(\frac{1}{2}-\frac{p}{2}+2p)},-2^{p},2^{(-\frac{1}{2}+\frac{p}{2})},0))\,,
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&-\frac{1}{2}\log(1+2^{(\frac{1}{2}-\frac{p}{2})}+2^{-p}+2^{(\frac{1}{2}+\frac{p}{2}-2p)}+2^{-2p}) \\
&=\frac{1}{2^{12p}}P\left(1,2^{12p},24,\left(-2^{(-\frac{1}{2}+\frac{p}{2}+11p)},0,\ -2^{(\frac{1}{2}-\frac{p}{2}+11p)},2^{10p},\right.\right. \\
&2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+8p)},-2^{8p},2^{(\frac{1}{2}-\frac{p}{2}+8p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+6p)}, \\
&-2^{1+6p},2^{(-\frac{1}{2}+\frac{p}{2}+5p)},0,2^{(\frac{1}{2}-\frac{p}{2}+5p)},-2^{4p},-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0, \\
&\left.\left.2^{(-\frac{1}{2}+\frac{p}{2}+2p)},2^{2p},-2^{(\frac{1}{2}-\frac{p}{2}+2p)},0,-2^{(-\frac{1}{2}+\frac{p}{2})},2\right)\right)\ , \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\sqrt{3}}{2}\log\left[\frac{1+2^{-\frac{1}{2}-\frac{p}{2}}(1+\sqrt{3})+2^{-p}}{1+2^{-\frac{1}{2}-\frac{p}{2}}(1-\sqrt{3})+2^{-p}}\right] \\
&=\frac{3}{2^{12p}}P\left(1,2^{12p},24,\left(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},-2^{11p},\ 0,0,\right.\right. \\
&2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+8p)},0,0,2^{7p},-2^{(-\frac{1}{2}+\frac{p}{2}+6p)}, \\
&\left.\left.0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},2^{5p},0,0,-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0,\ 2^{(-\frac{1}{2}+\frac{p}{2}+2p)},0,0,-2^{p},2^{(-\frac{1}{2}+\frac{p}{2})},0\right)\right) \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
&\sqrt{3}\arctan\left[\left(\frac{1+2^{\frac{p+1}{2}}}{-1+2^{\frac{p+1}{2}}+2^{p+1}}\right)\sqrt{3}\right] \\
&=\frac{3}{2^{12p}}P\left(1,2^{12p},24,\left(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},0,\ 0,-2^{10p},\right.\right. \\
&2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+8p)},-2^{8p},0,0,2^{(-\frac{1}{2}+\frac{p}{2}+6p)}, \\
&0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},0,0,2^{4p},-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0, \\
&\left.\left.-2^{(-\frac{1}{2}+\frac{p}{2}+2p)},2^{2p},0,0,-2^{(-\frac{1}{2}+\frac{p}{2})},0\right)\right)\ , \qquad (21)
\end{aligned}
$$

and

$$
\begin{aligned}
&\arctan\left[\frac{2^{\frac{1-p}{2}}+1}{2^{\frac{1+p}{2}}+1}\right] \\
&=\frac{1}{2^{12p}}P\left(1,2^{12p},24,\left(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},2^{11p},\ -2^{(\frac{1}{2}-\frac{p}{2}+11p)},0,\right.\right. \\
&-2^{(-\frac{1}{2}+\frac{p}{2}+9p)},2^{1+9p},-2^{(-\frac{1}{2}+\frac{p}{2}+8p)},0,-2^{(\frac{1}{2}-\frac{p}{2}+8p)},2^{7p},2^{(-\frac{1}{2}+\frac{p}{2}+6p)}, \\
&0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},-2^{5p},2^{(\frac{1}{2}-\frac{p}{2}+5p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+3p)},-2^{1+3p}, \\
&\left.\left.2^{(-\frac{1}{2}+\frac{p}{2}+2p)},0,2^{(\frac{1}{2}-\frac{p}{2}+2p)},-2^{p},-2^{(-\frac{1}{2}+\frac{p}{2})},0\right)\right)\ . \qquad (22)
\end{aligned}
$$

Particular cases of these formulas will be discussed in section 6.1.

5 Degree 2 formulas

The imaginary part of the dilogarithm function can be expressed in closed form as [6]

$$\operatorname{Im}\operatorname{Li}_2\left[qe^{ix}\right] = \omega\log q + \frac{1}{2}\operatorname{Cl}_2(2\omega) - \frac{1}{2}\operatorname{Cl}_2(2\omega+2x) + \frac{1}{2}\operatorname{Cl}_2(2x)\,, \tag{23}$$

where

$$\omega = \arctan\left(\frac{q\sin x}{1-q\cos x}\right).$$

Using Eq. (23), the results Eq. (9) and Eq. (10) can be written in degree 2 as

$$\begin{aligned}
&-(\omega_1+\omega_2)p\log 2 + \operatorname{Cl}_2(2\omega_1) + \operatorname{Cl}_2(2\omega_2)\\
&-\operatorname{Cl}_2(2\omega_1+\pi/6) + \operatorname{Cl}_2(-2\omega_2+5\pi/6) + 1/2\operatorname{Cl}_2(\pi/3)\\
&= 2\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{i\pi}{12}\right)\right] + 2\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{7i\pi}{12}\right)\right]\\
&= \frac{\sqrt{3}}{2^{12p-1}}P(2,2^{12p},24,(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},0,0,2^{10p},\\
&2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+8p)},2^{8p},0,0,2^{(-\frac{1}{2}+\frac{p}{2}+6p)},\\
&0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},0,0,-2^{4p},-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0,\\
&-2^{(-\frac{1}{2}+\frac{p}{2}+2p)},-2^{2p},0,0,-2^{(-\frac{1}{2}+\frac{p}{2})},0))
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
&(\omega_2-\omega_1)p\log 2 + \operatorname{Cl}_2(2\omega_1) - \operatorname{Cl}_2(2\omega_2)\\
&-\operatorname{Cl}_2(2\omega_1+\pi/6) - \operatorname{Cl}_2(-2\omega_2+5\pi/6) + 4G/3\\
&= 2\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{i\pi}{12}\right)\right] - 2\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{7i\pi}{12}\right)\right]\\
&= \frac{1}{2^{12p-1}}P(2,2^{12p},24,(-2^{(-\frac{1}{2}+\frac{p}{2}+11p)},2^{11p},2^{(\frac{1}{2}-\frac{p}{2}+11p)},0,\\
&2^{(-\frac{1}{2}+\frac{p}{2}+9p)},2^{1+9p},2^{(-\frac{1}{2}+\frac{p}{2}+8p)},0,2^{(\frac{1}{2}-\frac{p}{2}+8p)},2^{7p},-2^{(-\frac{1}{2}+\frac{p}{2}+6p)},\\
&0,2^{(-\frac{1}{2}+\frac{p}{2}+5p)},-2^{5p},-2^{(\frac{1}{2}-\frac{p}{2}+5p)},0,-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},-2^{1+3p},\\
&-2^{(-\frac{1}{2}+\frac{p}{2}+2p)},0,-2^{(\frac{1}{2}-\frac{p}{2}+2p)},-2^{p},2^{(-\frac{1}{2}+\frac{p}{2})},0))\,,
\end{aligned} \tag{25}$$

where ω_1 and ω_2 are given by

$$\tan\omega_1 = \frac{\sqrt{3}-1}{\sqrt{2}^{p+3}-\sqrt{3}-1}$$

and

$$\tan\omega_2 = \frac{\sqrt{3}+1}{\sqrt{2}^{p+3}+\sqrt{3}-1}\,.$$

Similarly, using Eq. (23), the results Eq. (13) and Eq. (14) can be written in degree 2 as

$$\begin{aligned}
&-(\omega_3+\omega_4)p\log 2+\mathrm{Cl}_2(2\omega_3)+\mathrm{Cl}_2(2\omega_4)\\
&-\mathrm{Cl}_2(2\omega_3+5\pi/6)+\mathrm{Cl}_2(-2\omega_4+\pi/6)-1/2\mathrm{Cl}_2(\pi/3)\\
&=2\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{5i\pi}{12}\right)\right]+2\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{11i\pi}{12}\right)\right]\\
&=\frac{\sqrt{3}}{2^{12p-1}}P\left(2,2^{12p},24,\left(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},0,\ 0,-2^{10p},\right.\right.\\
&2^{(-\frac{1}{2}+\frac{p}{2}+9p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+8p)},-2^{8p},0,0,2^{(-\frac{1}{2}+\frac{p}{2}+6p)},\\
&0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},0,0,2^{4p},-2^{(-\frac{1}{2}+\frac{p}{2}+3p)},0,\\
&\left.\left.-2^{(-\frac{1}{2}+\frac{p}{2}+2p)},2^{2p},0,0,-2^{(-\frac{1}{2}+\frac{p}{2})},0\right)\right)
\end{aligned}\tag{26}$$

and

$$\begin{aligned}
&(\omega_4-\omega_3)p\log 2+\mathrm{Cl}_2(2\omega_3)-\mathrm{Cl}_2(2\omega_4)\\
&-\mathrm{Cl}_2(2\omega_3+5\pi/6)-\mathrm{Cl}_2(-2\omega_4+\pi/6)+4G/3\\
&=2\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{5i\pi}{12}\right)\right]-2\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}^p}\exp\left(\frac{11i\pi}{12}\right)\right]\\
&=\frac{1}{2^{12p-1}}P\left(2,2^{12p},24,\left(2^{(-\frac{1}{2}+\frac{p}{2}+11p)},2^{11p},\ -2^{(\frac{1}{2}-\frac{p}{2}+11p)},0,\right.\right.\\
&-2^{(-\frac{1}{2}+\frac{p}{2}+9p)},2^{1+9p},-2^{(-\frac{1}{2}+\frac{p}{2}+8p)},0,-2^{(\frac{1}{2}-\frac{p}{2}+8p)},2^{7p},2^{(-\frac{1}{2}+\frac{p}{2}+6p)},\\
&0,-2^{(-\frac{1}{2}+\frac{p}{2}+5p)},-2^{5p},2^{(\frac{1}{2}-\frac{p}{2}+5p)},0,2^{(-\frac{1}{2}+\frac{p}{2}+3p)},-2^{1+3p},\\
&\left.\left.2^{(-\frac{1}{2}+\frac{p}{2}+2p)},0,2^{(\frac{1}{2}-\frac{p}{2}+2p)},-2^{p},-2^{(-\frac{1}{2}+\frac{p}{2})},0\right)\right)\ ,
\end{aligned}\tag{27}$$

where ω_3 and ω_4 are given by

$$\tan\omega_3=\frac{\sqrt{3}+1}{\sqrt{2}^{p+3}-\sqrt{3}+1}$$

and

$$\tan\omega_4=\frac{\sqrt{3}-1}{\sqrt{2}^{p+3}+\sqrt{3}+1}\ .$$

In the above formulas, $\mathrm{G}=\mathrm{Cl}_2(\pi/2)$ is Catalan's constant.

In deriving Eqs. (26) and (27), we used Eq.4.32 pg 106 and Eq.4.17 pg 104 of [6], that is

$$\mathrm{Cl}_2\left(\frac{\pi}{6}\right)+\mathrm{Cl}_2\left(\frac{5\pi}{6}\right)=\frac{4\mathrm{G}}{3}$$

and

$$\mathrm{Cl}_2\left(x\right)-\mathrm{Cl}_2\left(\pi-x\right)=\frac{1}{2}\mathrm{Cl}_2\left(2x\right)\ .$$

6 Interesting particular cases

6.1 Degree 1 binary BBP-type formulas

We first note that Eqs. (15) and (19) give an infinite set of primes whose logarithms have binary BBP-type formulas, which serve to augment the known ones (e.g. those found in [2]) and [5]. Similarly, Eqs. (18) and (22) give an infinite set of rationals whose arctangents have binary BBP-type representations. We now present a couple of binary degree 1 formulas.

6.1.1 Binary BBP-type formula for $\log 2$

The identity

$$\log 2 = \operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right] + \operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{7i\pi}{12}\right)\right] \tag{28}$$

and $p = 1$ in Eq. (15) lead to the binary BBP-type formula

$$\begin{aligned}\log 2 \quad = \quad & \frac{1}{2^{12}}P(1, 2^{12}, 24, (2^{11}, 0, 2^{11}, 2^{10}, -2^9, 0, 2^8, -2^8, \\ & -2^8, 0, -2^6, -2^7, -2^5, 0, -2^5, -2^4, 2^3, 0, -2^2, 2^2, 2^2, 0, 1, 2))\end{aligned} \tag{29}$$

6.1.2 Binary BBP-type formula for $\pi\sqrt{3}$

The identity

$$\frac{\pi}{3} = \operatorname{Im}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right] + \operatorname{Im}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{7i\pi}{12}\right)\right] \tag{30}$$

and $p = 1$ in Eq. (17) lead to the binary BBP-type formula

$$\begin{aligned}\pi\sqrt{3} \quad = \quad & \frac{9}{2^{12}}P(1, 2^{12}, 24, (2^{11}, 0, 0, 2^{10}, 2^9, 0, 2^8, 2^8, 0, 0, \\ & 2^6, 0, -2^5, 0, 0, -2^4, -2^3, 0, -2^2, -2^2, 0, 0, -1, 0))\end{aligned} \tag{31}$$

6.1.3 Binary BBP-type formula for $\sqrt{3}\log(2 + \sqrt{3})$

The identity

$$\log(2 + \sqrt{3}) = \operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right] - \operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{7i\pi}{12}\right)\right] \tag{32}$$

and $p = 1$ in Eq. (16) lead to the binary BBP-type formula

$$\begin{aligned}\sqrt{3}\,\log(2 + \sqrt{3}) \quad = \quad & \frac{3}{2^{12}}P(1, 2^{12}, 24, (2^{11}, 2^{11}, 0, 0, 2^9, 0, -2^8, \\ & 0, 0, -2^7, -2^6, 0, -2^5, -2^5, 0, 0, -2^3, 0, 2^2, 0, 0, 2, 1, 0))\end{aligned} \tag{33}$$

6.1.4 Binary BBP-type formula for $\arctan 1/6$

The identity

$$-\arctan\left(\frac{1}{6}\right) = \operatorname{Im}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}^3}\exp\left(\frac{i\pi}{12}\right)\right] - \operatorname{Im}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}^3}\exp\left(\frac{7i\pi}{12}\right)\right] \tag{34}$$

and $p = 3$ in Eq. (18) lead to the binary BBP-type formula

$$\begin{aligned}\arctan\left(\frac{1}{6}\right) &= \frac{1}{2^{35}}P(1, 2^{36}, 24, (2^{33}, -2^{32}, -2^{31}, 0, -2^{27}, \\ &\quad -2^{27}, -2^{24}, 0, -2^{22}, -2^{20}, 2^{18}, 0, -2^{15}, 2^{14}, \\ &\quad 2^{13}, 0, 2^9, 2^9, 2^6, 0, 2^4, 2^2, -1, 0))\end{aligned} \tag{35}$$

6.2 Degree 2 binary BBP-type formulas

When $p = 1$ then $\omega_1 = \omega_2 = \pi/6$ and Eq. (26) and Eq. (27) simplify to

$$-\frac{\pi}{3}\log 2 + \frac{5}{2}\operatorname{Cl}_2\left(\frac{\pi}{3}\right) = 2\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right] + 2\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{7i\pi}{12}\right)\right] \tag{36}$$

and

$$\mathrm{G} = 3\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i7\pi}{12}\right)\right] - 3\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right], \tag{37}$$

respectively.

Choosing $q = 1/2$ and $x = \pi/3$ in Eq. (23) gives

$$-\pi\log 2 + 5\operatorname{Cl}_2\left(\frac{\pi}{3}\right) = 6\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi}{3}\right)\right]. \tag{38}$$

Note that

$$\begin{aligned}\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{3}\right)\right] &= \sum_{k=1}^{\infty}\frac{1}{2^k}\frac{\sin(k\pi/3)}{k^2} \\ &= \frac{\sqrt{3}}{2^{10}}P(2, 2^{12}, 24, (0, 2^{10}, 0, 2^9, 0, 0, 0, -2^7, 0, -2^6, 0, 0, 0, \\ &\quad 2^4, 0, 2^3, 0, 0, 0, -2, 0, -1, 0, 0))\end{aligned} \tag{39}$$

6.2.1 Binary BBP-type formula for $\sqrt{3}\operatorname{Cl}_2(\pi/3)$

Eliminating $\pi\log 2$ between Eqs. (36) and (38) we have

$$\begin{aligned}\operatorname{Cl}_2\left(\frac{\pi}{3}\right) &= \frac{12}{5}\left\{\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right] + \operatorname{Im}\operatorname{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{7i\pi}{12}\right)\right]\right\} \\ &\quad -\frac{12}{5}\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{3}\right)\right]\end{aligned} \tag{40}$$

Using Eqs. (26) (with $p = 1$) and (39) in Eq. (40), we obtain a binary BBP-type formula for $\sqrt{3}\,\mathrm{Cl}_2(\pi/3)$:

$$\begin{aligned}\sqrt{3}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) &= \frac{9}{5\cdot 2^{10}}\,P(2,2^{12},24,(2^{11},-2^{12},0,-2^{10},2^9,0,2^8,3\cdot 2^8,0,2^8,2^6,\\ &\qquad 0,-2^5,-2^6,0,-3\cdot 2^4,-2^3,0,-2^2,2^2,0,2^2,-1,0))\end{aligned} \tag{41}$$

6.2.2 Binary BBP-type formula for $\pi\sqrt{3}\log 2$

Eliminating $\mathrm{Cl}_2(\pi/3)$ between Eqs. (36) and (38) we have

$$\begin{aligned}\pi\log 2 &= 12\left\{\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right]+\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{7i\pi}{12}\right)\right]\right\}\\ &\quad -18\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{3}\right)\right]\end{aligned} \tag{42}$$

Using Eqs. (26) (with $p = 1$) and (39) in Eq. (42), we obtain a binary BBP-type formula for $\pi\sqrt{3}\log 2$:

$$\begin{aligned}\pi\sqrt{3}\log 2 &= \frac{9}{2^{10}}P(2,2^{12},24,(2^{11},-3\cdot 2^{11},0,-2^{11},2^9,0,2^8,2^{10},0,3\cdot 2^7,2^6,0,\\ &\quad -2^5,-3\cdot 2^5,0,-2^6,-2^3,0,-2^2,2^3,0,6,-1,0)\end{aligned} \tag{43}$$

It is interesting to remark that a variant of formula (43) (formula 27, section 5, in the BBP-Compendium) was discovered experimentally over ten years ago, but is hitherto unproved. Formula 27 in the Compendium is now proved by adding rational multiples of two zero relations to Eq. (43) (See [2]).

6.2.3 Binary BBP-type formula for Catalan's constant G

Eq. (37) leads immediately to

$$\begin{aligned}\mathrm{G} &= \frac{3}{2^{12}}\,P(2,2^{12},24,(2^{11},-2^{11},-2^{11},0,-2^9,-2^{10},-2^8,0,\\ &\qquad -2^8,-2^7,2^6,0,-2^5,2^5,2^5,0,2^3,2^4,2^2,0,2^2,2,1,0))\end{aligned} \tag{44}$$

6.3 Binary zero relations

The identity

$$\frac{\pi}{6} = \mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{2}\exp\left(\frac{i\pi}{3}\right)\right] = \sum_{k=1}^{\infty}\frac{1}{2^k}\frac{\sin(k\pi/3)}{k} \tag{45}$$

leads to the binary BBP-type formula

$$\pi\sqrt{3} = \frac{9}{2^{10}} P(1, 2^{12}, 24, (0, 2^{10}, 0, 2^9, 0, 0, 0, -2^7, 0, -2^6, 0, 0, 0, 2^4, 0, 2^3, 0, 0, 0, -2, 0, -1, 0, 0)) \quad (46)$$

Note that this is the base 2^{12} version of the formula listed for $\pi\sqrt{3}$ in section 4 of the BBP Compendium [2].

Combining Eqs. (30) and (45), we have the identity

$$0 = \operatorname{Im}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right] + \operatorname{Im}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{7i\pi}{12}\right)\right] - 2\operatorname{Im}\operatorname{Li}_1\left[\frac{1}{2}\exp\left(\frac{i\pi}{3}\right)\right] \quad (47)$$

which, with the use of (31) and (46) gives the binary zero relation

$$0 = P(1, 2^{12}, 24, (2^{11}, -2^{12}, 0, -2^{10}, 2^9, 0, 2^8, 3\cdot 2^8, 0, 2^8, 2^6, 0, -2^5, -2^6, 0, -3\cdot 2^4, -2^3, 0, -2^2, 2^2, 0, 2^2, -1, 0)) \quad (48)$$

Eq. (28) and the formula $\operatorname{Li}_1(1/2) = \log 2$ give the identity

$$0 = \operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right] + \operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{7i\pi}{12}\right)\right] - \operatorname{Li}_1\left[\frac{1}{2}\right] \quad (49)$$

which leads to the binary BBP-type zero relation

$$0 = P(1, 2^{12}, 24, (2^{11}, -2^{12}, 2^{11}, -2^{10}, -2^9, -2^{10}, 2^8, -3\cdot 2^8, -2^8, -2^8, -2^6, -2^8, -2^5, -2^6, -2^5, -3\cdot 2^4, 2^3, -2^4, -2^2, -2^2, 2^2, -2^2, 1, 0)) \quad (50)$$

The identity

$$0 = \operatorname{Im}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right] - \operatorname{Im}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{7i\pi}{12}\right)\right] \quad (51)$$

and $p = 1$ in Eq. (18) lead to the binary BBP-type zero relation

$$0 = P(1, 2^{12}, 24, (-2^{11}, 2^{11}, 2^{11}, 0, 2^9, 2^{10}, 2^8, 0, 2^8, 2^7, -2^6, 0, 2^5, -2^5, -2^5, 0, -2^3, -2^4, -2^2, 0, -2^2, -2, 1, 0)) \quad (52)$$

The identity

$$0 = \operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right] + \operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{7i\pi}{12}\right)\right] - 2\operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \quad (53)$$

leads to the binary zero relation

$$0 = P(1, 2^{12}, 24, (2^{11}, 0, -2^{12}, -3 \cdot 2^{10}, -2^9, 0, 2^8, 3 \cdot 2^8, 2^9, 0, -2^6, 0, -2^5, 0, 2^6, 3 \cdot 2^4, 2^3, 0, -2^2, -3 \cdot 2^2, -2^3, 0, 1, 0)) \tag{54}$$

The identity

$$\begin{aligned} 0 &= \operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{12}\right)\right] + \operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{7i\pi}{12}\right)\right] \\ &\quad +2\operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{2}\right)\right] - 2\operatorname{Re}\operatorname{Li}_1\left[\frac{1}{2}\exp\left(\frac{i\pi}{3}\right)\right] \end{aligned} \tag{55}$$

leads to the binary zero relation

$$0 = P(1, 2^{12}, 24, (2^{11}, -2^{13}, 2^{11}, 5 \cdot 2^{10}, -2^9, 2^{10}, 2^8, 3 \cdot 2^8, -2^8, -2^9, -2^6, -2^8, -2^5, -2^7, -2^5, 3 \cdot 2^4, 2^3, 2^4, -2^2, 5 \cdot 2^2, 2^2, -2^3, 1, 0)) \tag{56}$$

More degree 1 binary BBP-type zero relations in base 2^{12} and other bases can also be found in [7].

7 Conclusion

Using a clear and straightforward approach, explicit digit extraction BBP-type formulas in very general binary bases were discovered. As particular examples, new binary formulas were obtained for $\pi\sqrt{3}$, $\sqrt{3}\pi\log 2$ and some other polylogaritm constants. New binary BBP-type zero relations were also established.

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