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Degree sequence of configuration model with vertex faults

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Abstract: We study the degree sequence of configuration model of random graphs with random vertex deletion. The degree sequences are characterized under various deletion probabilities. Our results have implications in communication networks where random faults due to inner consumption and outer disturbance often occur.
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1 Introduction

Fix $n \in \mathbb{N}$. Let $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ be the degree sequence of a graph. The configuration model of random graphs [4, 7] can be constructed as follows. A configuration consists of n boxes, the j-th box of which contains d_j balls, and a perfect matching of the $\sum_{j=1}^{n} d_j$ balls is chosen uniformly at random. The edges of the perfect matching are called pairs. Here we assume the boxes are labeled $1, 2, \cdots, n$ and that the balls in the j-th box are labeled $1, 2, \cdots, d_j$. Denote this probability space by $\mathcal{C}_{n,\mathbf{d}}$. Given a configuration $C \in \mathcal{C}_{n,\mathbf{d}}$ we get a graph $G(C)$ by shrinking each box down to a vertex. The probability that $G(C)$ is simple is bounded below by a positive constant, and furthermore, it is uniformly distributed conditioned on being simple [1]. Hence, if an event is a.a.s. true for $G(C)$ when $C \in \mathcal{C}_{n,\mathbf{d}}$, then it is also a.a.s. true conditional on the event that $G(C)$ is simple.

In this paper, all asymptotics are as $n \to \infty$. We say that an event holds asymptotically almost surely (a.a.s.) if the probability that it holds tends to 1 following the convention in [4]. Our main interest in the current work concerns with the random vertex deletion process or vertex faults on $\mathcal{C}_{n,\mathbf{d}}$. This is motivated by the applications to communication networks, where nodes are often subject to random disturbances [3, 5, 6]. The formal box deletion process for configurations is described as follows. Given $C \in \mathcal{C}_{n,\mathbf{d}}$, form a new configuration $\hat{C}$ by independently deleting each box with probability p_n. Specifically:

- choose a random subset S of boxes such that for every box b, $b \in S$ independently with probability p_n; delete all boxes in S as well as balls adjacent to balls belonging to S by pairs.
- relabel the surviving boxes with the labels $1, 2, \cdots$, preserving the relative ordering of the boxes; relabel the balls within each surviving box in the same way.

Our main result, presented in the next section, is a characterization of the degree sequence of configuration model $\mathcal{C}_{n,\mathbf{d}}$ under the aforementioned vertex faults. To the best of our knowledge, the vertex deletion process for general configuration model analyzed in this paper does not appear in the literature. A related work in [3] deals with random regular graph, which can be viewed as a configuration model with constant degree sequences.

2 The results

Let $E(X)$ and $Var(X)$ be the mean and variance of a random variable X, respectively. Suppose that the degree sequence $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ satisfies $d_i = ((1+o(1))d$ for $i = 1, 2, \cdots, n$, and that the deletion probability p_n satisfies $np_n \to \infty$ and $np_n(1-p_n) \to 0$. Let $C \in \mathcal{C}_{n,\mathbf{d}}$ and let S be the random set of boxes chosen for deletion. Denote by $s = |S|$, the number of boxes in S. By virtue of the sharp concentration of binomial random variables (see e.g. [2] pp. 25), we obtain

$$s \sim np_n$$

a.a.s., as $n \to \infty$.

Let $\hat{C}$ be the result of deleting the boxes in S from C and performing the necessary relabelings of boxes and balls. Thus, $\hat{C}$ contains $n - s$ boxes. For $0 \leq j \leq d$, let N_j denote the number of boxes in $\hat{C}$ with j balls. Let

$$\mu_j = n\binom{d}{j} p_n^{d-j}.$$

Our main result is the following.

Proposition 1. *Construct $\hat{C}$ from $C \in \mathcal{C}_{n,\mathbf{d}}$ by deleting the boxes in S as described in Section 1. For $0 \leq j \leq d$, we obtain $EN_j \sim \mu_j$ and a.a.s.*

$$N_j = \begin{cases} (1+o(1))\mu_j & if\ \mu_j \to \infty, \\ O(\omega_n) & if\ \mu_j = O(1), \\ 0 & if\ \mu_j = o(1). \end{cases}$$

where $\omega_n \to \infty$ arbitrarily slowly, as $n \to \infty$. Moreover, in all cases, a.a.s. $N_j = o(\mu_l)$ for $0 \leq j < l \leq d$.

To prove Proposition 1, the following lemma is useful.

Lemma 1.([1] pp. 54) *Let k be a fixed positive integer and let $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ be a degree sequence satisfying $0 \leq d_i \leq d$ for all i. Then the probability that a random element of $\mathcal{C}_{n,\mathbf{d}}$ contains k specified pairs is $(1+o(1))(2m)^{-k}$, where $m = (d_1 + d_2 + \cdots + d_n)/2$ is the number of pairs in the configuration.*

Proof of Proposition 1. Fix $n \in \mathbb{N}$ and $0 \le j \le d$. Choose a random configuration $C \in \mathcal{C}_{n,\mathbf{d}}$. By Lemma 1, the probability that a given box $b \notin S$ is incident with exactly $d-j$ pairs which are incident with balls in S is asymptotically equal to

$$\begin{aligned}\binom{d}{j}\binom{ds}{d-j}(d-j)!(dn)^{-(d-j)} &\sim \binom{d}{j}s^{d-j}n^{-(d-j)}\\ &\sim \binom{d}{j}p_n^{d-j}\\ &= \frac{\mu_j}{n}.\end{aligned}$$

Hence, we obtain $EN_j \sim \mu_j$ by using the linearity of expectation and $s/n = o(1)$.

Suppose that $\mu_j \to \infty$. Similar calculations for an ordered pair of boxes $b, c \notin S$ show that $E(N_j(N_j-1)) \sim (EN_j)^2$. Therefore, $Var(N_j) \sim EN_j \sim \mu_j$. By Chebyshev's inequality, for $\varepsilon \ll 1/\sqrt{\mu_j}$, we obtain

$$\begin{aligned}P\left(\left|\frac{N_j}{\mu_j} - \frac{EN_j}{\mu_j}\right| \ge \varepsilon\right) &\le \frac{Var\left(\frac{N_j}{\mu_j}\right)}{\varepsilon^2}\\ &\sim \frac{EN_j}{\mu_j^2\varepsilon^2}\\ &\sim \frac{1}{\mu_j\varepsilon^2} \to 0.\end{aligned}$$

Consequently, a.a.s. $N_j \sim \mu_j$. When $\mu_j = O(1)$, by Markovian inequality we have

$$P(N_j \ge \omega_n) \le \frac{\mu_j}{\omega_n} \to 0$$

for any $\omega_n \to \infty$. Therefore, $N_j = O(\omega_n)$ a.a.s. Likewise, we get

$$P(N_j \ge 1) \le \mu_j \to 0$$

for $\mu_j = o(1)$, and thus $N_j = 0$ a.a.s.

Finally, for $\varepsilon \to 0$ sufficiently slowly, by Markovian inequality, we derive that

$$P\left(\frac{N_j}{\mu_l} \ge \varepsilon\right) \le \frac{\mu_j}{\mu_l\varepsilon} = \Theta\left(\frac{p_n^{l-j}}{\varepsilon}\right)$$

which tends to zero for $j < l$. Accordingly, we get $N_j = o(\mu_l)$ a.a.s. □

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