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Some properties of unitary addition Cayley graphs

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Abstract: Let Γ be an abelian group and B be a subset of Γ. The *addition Cayley graph* $G' = Cay^+(\Gamma, B)$ is the graph having the vertex set $V(G') = \Gamma$ and the edge set $E(G') = \{ab : a + b \in B\}$, where $a, b \in \Gamma$. For a positive integer $n > 1$, the *unitary addition Cayley graph* G_n is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes set of all units of the ring Z_n, then two vertices a, b are adjacent if and only if $a + b \in U_n$. The unitary addition Cayley graph G_n is also defined as, $G_n = Cay^+(Z_n, U_n)$. In this paper, we discuss the several properties of unitary addition Cayley graphs and also obtain the characterization of planarity and outerplanarity of unitary addition Cayley graphs.
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1 Introduction

For standard terminology and notation in graph theory we refer Harary [17] and West [25]. We look up Rose [23] for number theoretic properties. A *graph* G is a pair (V, E), where V is a non-empty finite set, and E is a set of unordered pairs of elements of V. The elements of V are called the *vertices* of G, and the elements of E are the *edges* of G. The set of vertices and edges of a graph G is denoted by $V(G)$ and $E(G)$, respectively. The cardinality of $V(G)$ is called the *order* of G. $|V(G)|$ and $|E(G)|$ denote the cardinality of $V(G)$ and $E(G)$, respectively. If $|V(G)| = p$ and $|E(G)| = q$, then G is called a (p, q)-graph.

The *degree* of a vertex v, $d(v)$ in G is the number of edges incident at v. If degree of each vertex is equal, say r in G, then G is called *r-regular graph.* A graph is called (r_1, r_2)-*semiregular* if its vertex set can be partitioned into two subsets V_1 and V_2 such that all the vertices in V_i are of degree r_i for $i = 1, 2$. G is called *trivial graph* if $E(G)$ is empty set and it contains exactly one vertex. Vertices u and v of a graph G are *adjacent* if $uv \in E(G)$. If G has n number of vertices and each pair of vertices is adjacent, then it is called *complete graph* K_n of order n. Throughout the text, we consider non-trivial, finite, undirected graphs with no loops or multiple edges.

A *walk* in a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2,$ $..., e_n, v_n$, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. It is *closed* if $v_0 = v_n$. A closed walk in which all the vertices are distinct is called *cycle*. It is written as $C = (v_1, v_2, \ldots, v_n, v_1)$. A closed walk in which all the edges are distinct is called *circuit*. An *Eulerian circuit* in a graph G is a circuit containing every edge of G and G is an *Eulerian graph* if it contains an Eulerian circuit. A *Hamiltonian cycle* in a graph G is a cycle containing every vertex of G and G is called a *Hamiltonian graph* if it contains a Hamiltonian cycle.

A graph G is embedded in a surface S when its vertices are represented by points in S, and each edge by a curve joining corresponding points in S, in such a way that no curve intersects itself, and two curves intersect each other only at a common vertex. A graph which can be embedded in the plane is called *planar*. A planar graph is called *outerplanar* if it can be embedded in the plane in such a way that all of its vertices are in the same face.

Let Γ be a group and B be a subset of Γ such that B does not contain identity of Γ. Assume $B^{-1} = \{b^{-1} : b \in B\} = B$. The *Cayley graph* $X' = Cay(\Gamma, B)$ is an undirected graph having vertex set $V(X') = \Gamma$ and edge set $E(X') = \{ab : ab^{-1} \in B\}$, where $a, b \in \Gamma$. The Cayley graph X' is a regular graph of degree $|B|$. Its connected components are the right cosets of the subgroup generated by B. Therefore, if B generates Γ, then X' is a connected graph. The books on algebraic graph theory by Biggs [5] and by Godsil & Royle [13] provide many information regarding Cayley graphs.

For a positive integer $n > 1$, the *unitary Cayley graph* X_n is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes set of all units of the ring Z_n, then two vertices a, b are adjacent if and only if $a - b \in U_n$. The unitary Cayley graph X_n is also defined as, $X_n = Cay(Z_n, U_n)$. The structure and various properties of unitary Cayley graphs have been studied in literature (*see* [1], [3], [4], [6], [9], [10], [11], [12], [18], [22], [24]).

Let Γ be an abelian group and B be a subset of Γ. The *addition Cayley graph* $G' = Cay^+(\Gamma, B)$ is the graph having the vertex set $V(G') = \Gamma$ and the edge set $E(G') = \{ab : a + b \in B\}$, where $a, b \in \Gamma$. Several properties of addition Cayley graphs have been discussed in literature (*see* [2], [7], [8], [14], [15], [16], [20], [21]).

Now, we introduce the unitary addition Cayley graphs as follows:

For a positive integer $n > 1$, the *unitary addition Cayley graph* G_n is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes set of all units of the ring Z_n, then two vertices a, b are adjacent if and only if $a + b \in U_n$. The unitary addition Cayley graph G_n is also defined as, $G_n = Cay^+(Z_n, U_n)$.

Some examples of unitary addition Cayley graphs are displayed in Figure 1. Throughout the text, we consider $n \geq 2$.

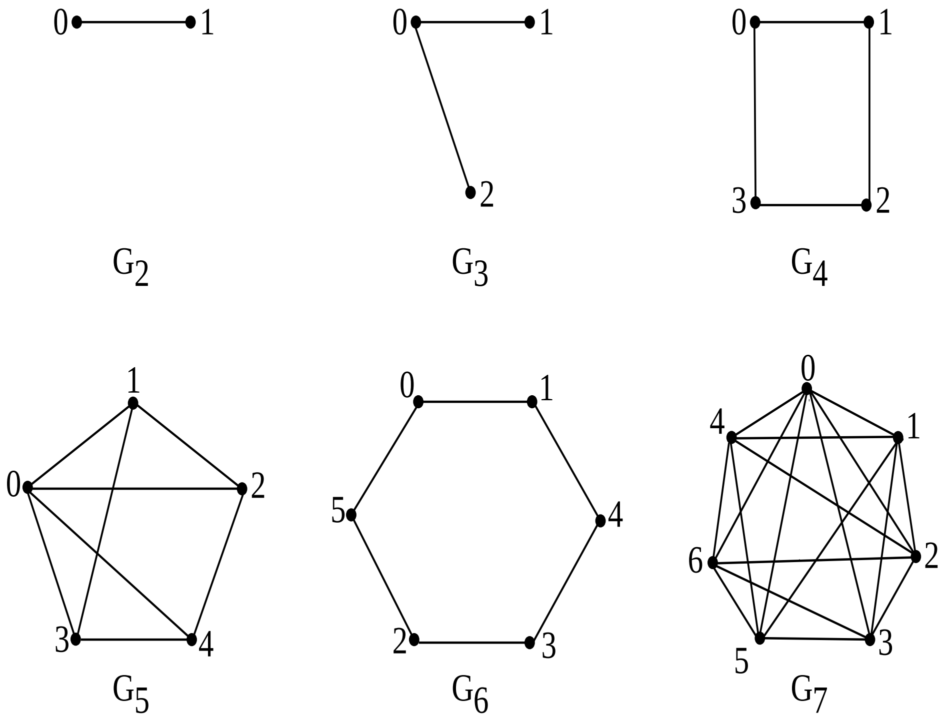

Figure 1: Some examples of unitary addition Cayley graphs.

Unitary addition Cayley graphs are motivated from the definition of addition Cayley graphs. Now, we study structure and several properties of unitary addition Cayley graphs.

2 Degree of vertices in unitary addition Cayley graphs

In what follows here onward, $\phi(n)$ denotes the well known Euler's totient function, which gives the number of numbers less than n and that are co-prime to n.

Theorem 1. *Let m be any vertex of the unitary addition Cayley graph G_n. Then,*

$$d(m) = \begin{cases} \phi(n) & \text{if } n \text{ is even}, \\ \phi(n) & \text{if } n \text{ is odd and } (m,n) \neq 1, \\ \phi(n) - 1 & \text{if } n \text{ is odd and } (m,n) = 1. \end{cases}$$

Proof. Suppose n is even and m is a vertex of the unitary addition Cayley graph G_n. Since $|U_n| = \phi(n)$, let $U_n = \{1, m_2, \ldots, m_{\phi(n)-1}, m_{\phi(n)}\}$. By the definition of unitary addition Cayley graph, $0 \in V(G_n)$ is adjacent to a vertex $m \in V(G_n)$ if and only if $(0 + m, n) = 1$. This implies, $(m, n) = 1$. That means, $m \in U_n$. It follows that $d(0) = \phi(n)$. Suppose $l \in U_n$, then $(l, n) = 1$. Since $(l - m + m, n) = 1$, the vertex m is adjacent to all the vertices of type $l-m$ in G_n. Since n is even, U_n does not contain an even number. That means, $2m \not\equiv l \pmod{n}$, which implies that $m \not\equiv l-m \pmod{n}$. Thus, all the vertices of type $l-m$ are distinct and not similar to the vertex m. Hence, $d(m) = \phi(n)$.

Next, suppose n is odd and $(m,n) \neq 1$. Then, $m \notin U_n$, which implies that $2m \notin U_n$. Thus, for $l \in U_n$, $m \not\equiv l-m \pmod{n}$. Hence, $d(m) = \phi(n)$.

Now, let n be odd and $(m,n) = 1$. Then, $m \in U_n$, which implies that $2m \in U_n$. Thus, for $l \in U_n$, $m \equiv l-m \pmod{n}$. But m cannot be adjacent to itself. Hence, $d(m) = \phi(n) - 1$. □

Corollary 2. *The total number of edges in the unitary addition Cayley graph G_n is*

$$\begin{cases} \frac{1}{2}n\phi(n) & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1)\phi(n) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose n is even, then $n\phi(n) = 2q$, where q is the total number of edges in G_n. It implies that, $q = \frac{n}{2}\phi(n)$. Next, suppose n is odd. Then,

$$(n - \phi(n))\phi(n) + (\phi(n) - 1)\phi(n) = 2q.$$

It implies that,

$$q = \frac{1}{2}(n-1)\phi(n).$$

This completes the proof.

□

Corollary 3. *The unitary addition Cayley graph G_n is $\phi(n)$-regular if n is even and $(\phi(n), \phi(n)-1)$-semiregular if n is odd.*

Theorem 4. *The girth of the unitary addition Cayley graph G_n*

$$= \begin{cases} 3 & \text{if } n \text{ is odd and } n > 3, \\ 4 & \text{if } n \text{ is even, } n > 2 \text{ and } n \not\equiv 0 \ (\text{mod } 3). \end{cases}$$

Proof. Suppose n is odd and $n > 3$, then we obtain a cycle

$$C' = (0, 2, n-1, 0)$$

of length three in G_n. Thus, the girth of G_n is equal to 3. Next, suppose n is even, $n > 2$ and $n \not\equiv 0 \pmod{3}$. Then, we get a cycle

$$C'' = (0, 1, 2, n-1, 0)$$

of length four in G_n. If possible, let G_n have a cycle of length three. Since this cycle contains three vertices, two of them must be either even or odd. As the addition of either two even or two odd numbers gives an even number, they cannot be adjacent in G_n. Thus, the girth of G_n is equal to 4. □

3 Relation between unitary addition Cayley graphs and unitary Cayley graphs

In this section, we establish a relation between unitary addition Cayley graphs and unitary Cayley graphs.

Theorem 5. *The unitary addition Cayley graph* G_n *is isomorphic to the unitary Cayley graph* X_n *if and only if* n *is even.*

Proof. **Necessity**: Suppose $G_n \cong X_n$. We shall show that n is even. If possible, let n be odd. Then due to Theorem 1, G_n has some vertices of degree $\phi(n) - 1$, but X_n does not have any vertex of degree $\phi(n) - 1$. That means, $G_n \ncong X_n$, a contradiction to our assumption. Hence, n is even.

Sufficiency: Suppose n is even. We shall show that $G_n \cong X_n$. Consider a function $f : V(G_n) \to V(X_n)$ such that

$$f(m) = \begin{cases} m & \text{if } m \text{ is even} \\ n - m & \text{if } m \text{ is odd.} \end{cases}$$

Let $U_n = \{a_1, a_2, \ldots, a_{\phi(n)}\}$. Since the vertex m is adjacent to the vertices of type $a_r - m$, consider a set

$$S_m = \{a_1 - m,\ a_2 - m, \ldots, a_{\phi(n)} - m\}.$$

Suppose two vertices i and j are adjacent in G_n, then j is of the form $a_r - i$, where $a_r \in U_n$.

Case I: If i is even, then $j = a_r - i$ is odd. This implies that $f(i) = i$ and $f(j) = n - j = n - (a_r - i)$. Now,

$$\begin{aligned} (i - (n - j),\ n) &= (i - (n - (a_r - i)),\ n) \\ &= (i - n + (a_r - i),\ n) \\ &= (i - n + a_r - i,\ n) \\ &= (-n + a_r,\ n) \\ &= (a_r,\ n) \\ &= 1. \end{aligned}$$

Thus, $f(i)$ and $f(j)$ are adjacent in X_n.

Case II: If i is odd, then $j = a_r - i$ is even. This implies that $f(i) = n - i$ and $f(j) = j = (a_r - i)$. Now,

$$((n - i) - j,\ n) = ((n - i) - (a_r - i),\ n)$$

$$= (n - i - a_r + i,\ n)$$
$$= (n - a_r,\ n)$$
$$= 1.$$

Thus, $f(i)$ and $f(j)$ are adjacent in X_n.

Now, one can easily verify that f is one-one and onto function and it preserves adjacency, also. Hence $G_n \cong X_n$.

□

Theorem 6. [18] *There is no self complementary unitary Cayley graph X_n for $n \geq 2$.*

Theorem 7. *There is no self-complementary unitary addition Cayley graph G_n for $n \geq 2$.*

Proof. Due to Theorem 5, if n is even, then $G_n \cong X_n$. Now using Theorem 6, G_n is not self-complementary graph. Next, suppose n is odd. If possible, G_n is self-complementary graph. That means, $G_n \cong \overline{G}_n$. Since $G_n \cup \overline{G}_n \cong K_n$,

$$\frac{n(n-1)}{2} = (n-1)\phi(n)$$
$$i.e.\ \ \phi(n) = \frac{n}{2}$$

It implies that, n is even, a contradiction to the assumption that n is odd. Hence the theorem. □

Theorem 8. [9] *The unitary Cayley graph X_n, $n \geq 2$, is bipartite if and only if n is even.*

Theorem 9. *The unitary addition Cayley graph G_n, $n \geq 2$, is bipartite if and only if either n is even or $n = 3$.*

Proof. **Necessity**: Suppose G_n, $n \geq 2$, is bipartite. We shall show that either n is even or $n = 3$. If possible, let n be odd and $n > 3$. Then the vertex 0 is adjacent to the vertices 2 and $n-1$, and the vertex $n-1$ is adjacent to the vertex 2 in G_n. That means, we obtain a cycle $C' = (0, 2, n-1, 0)$ of length three in G_n. This implies that, G_n is not bipartite, a contradiction to our assumption. Hence, either n is even or $n = 3$.

Sufficiency: Suppose n is even. Then due to Theorem 5 and Theorem 8, G_n, $n \geq 2$, is bipartite. Next, suppose $n = 3$. Then, G_3 is a tree and every tree is bipartite. It follows that G_3 is bipartite. Hence the theorem. □

Theorem 10. *If n is odd, then the unitary addition Cayley graph G_n is k-partite, where $k \leq (\frac{\phi(n)}{2} + r)$, r is the number of distinct prime factors of n and equality holds if n is a prime number.*

Proof. Suppose $U_n = \{1, m_2, \ldots, m_{\phi(n)-1}, n-1\}$. Since the vertex 1 is not adjacent to the vertex $n-1$ and the vertex m_i is not adjacent to the vertex $m_{\phi(n)-i+1}$ in G_n for $i = 2, 3, \ldots, \frac{\phi(n)}{2}$, we obtain $\frac{\phi(n)}{2}$ independent sets. Now, we are left with those elements of Z_n which are not co-prime to n. That means, these are multiples of prime factors of n. Let S_{p_i} denotes the set of all elements which are multiples of p_i. Clearly, each S_{p_i} is an independent set and the vertex 0 can be included in S_{p_1}. Thus, G_n is k-partite, where $k \leq (\frac{\phi(n)}{2} + r)$, r is the number of distinct prime factors of n and if n is a prime number, then equality holds. □

4 Traversability of unitary addition Cayley graphs

In this section, we consider the Eulerian and Hamiltonian properties of unitary addition Cayley graphs.

Theorem 11. *The unitary addition Cayley graph G_n is Hamiltonian, if*

(i) n *is even, or*

(ii) n *is a prime number and* $n > 3$*, or*

(iii) n *is odd and* $n \not\equiv 0 \ (mod\, 3)$.

Proof. Suppose n is even. Now, we construct the cycle,

$$C = (0, n-1, 2, n-3, 4, \ldots, n-4, 3, n-2, 1, 0).$$

Since the cycle C contains all the vertices of G_n exactly once, C is a Hamiltonian cycle of G_n. Thus, G_n is Hamiltonian. Next, suppose n is a prime number and $n > 3$. Now, we construct a cycle

$$C' = (n-2, n-4, \ldots, 3, 1, 0, 2, 4, 6, 8, \ldots, n-3, n-1, n-2).$$

Again the cycle C' contains all the vertices of G_n exactly once, implying that, C' is a Hamiltonian cycle of G_n. Thus, G_n is Hamiltonian. Next, let n be odd and $n \not\equiv 0 \pmod 3$. Then, there exists a cycle

$$C'' = (n-1, 0, n-2, 1, n-3, 2, \ldots, (\frac{n-3}{2}), (\frac{n-1}{2}), n-1).$$

The cycle C'' forms a spanning cycle of G_n. Thus, G_n is Hamiltonian. □

Theorem 12. *The unitary addition Cayley graph G_n is connected for all n.*

Proof. Due to Theorem 11, G_n is Hamiltonian graph for all n except when n is odd and multiple of 3. This implies that G_n is connected for every n, except when $n = 3m$. Then, we can, by the definition of G_n, find a hamiltonian path

$$P = (n-1, 0, n-2, 1, n-3, 2, \ldots, (\frac{n-1}{2})).$$

Thus, G_n is connected. □

Theorem 13. [25] *A graph* G *is Eulerian if and only if* G *is connected and its vertices all have even degree.*

Theorem 14. *The unitary addition Cayley graph* G_n *is Eulerian if and only if* n *is even and* $n > 2$.

Proof. Suppose G_n is Eulerian. If possible, suppose n is odd and greater than two. Then, due to Theorem 1, G_n has vertices of odd degree. Now, by the above mentioned Theorem 13, G_n is not Eulerian, a contradiction to our assumption. Next, suppose n is even and $n > 2$, then the degree of each vertex in G_n is $\phi(n)$ and $\phi(n)$ is even $\forall\, n \geq 3$. Therefore, the result follows by invoking the above Theorem 13. □

5 Planarity of unitary addition Cayley graphs

In this section, we characterize the planarity and outerplanarity of unitary addition Cayley graphs. Kuratowski [19] obtained the following characterization of planar graphs.

Theorem 15. *A graph is planar if and only if it does not have a subgraph homeomorphic to* K_5 *or* $K_{3,3}$.

Theorem 16. [6] *The unitary Cayley graph* X_n *is planar if and only if the value of* n *is* $1,\ 2,\ 3,\ 4,\ 5$ *or* 6.

Theorem 17. *The unitary addition Cayley graph* G_n *is planar if and only if the value of* n *is* $1,\ 2,\ 3,\ 4,\ 5$ *or* 6.

Proof. If $n \in \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$, then we can easily produce a planar representation of G_n. We shall now show that G_n is non-planar for any other value of n. Since a simple planar connected graph has a vertex of degree less than six, if $\phi(n) - 1 \geq 6$, G_n cannot be planar. This implies that G_n is non-planar for $n \geq 15$. Since $G_n \cong X_n$ for even values of n, due to Theorem 16, G_n is non-planar for $n = 8,\ 10,\ 14$. It remains to prove the result for $n = 7,\ 9,\ 11,\ 13$. For $n = 11$, $\phi(11) = 10$ and for $n = 13$, $\phi(13) = 12$. Therefore, G_n is non-planar for $n = 11, 13$. For $n = 7,\ 9$; G_n has subgraphs homeomorphic to K_5 and $K_{3,3}$, respectively (see Figure 2). Therefore, due to Theorem 15, G_n is non-planar for $n = 7,\ 9$. Hence the theorem. □

Theorem 18. [17] *A graph* G *is outerplanar if and only if it has no subgraph homeomorphic to* K_4 *or* $K_{2,3}$ *except* $K_4 - x$.

Theorem 19. *The unitary addition Cayley graph* G_n *is outerplanar if and only if the value of* n *is* $1,\ 2,\ 3,\ 4$ *or* 6.

Proof. Due to Theorem 17, the only possibility for G_n to be outerplanar is $n \in \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$. If $n \in \{1,\ 2,\ 3,\ 4,\ 6\}$, then we can easily produce an outerplanar representation of G_n. We shall now show that G_n is non-outerplanar for $n = 5$. Since G_5 has a subgraph homeomorphic to K_4 (see Figure 3), due to Theorem 18, G_5 is non-outerplanar. Hence the theorem. □

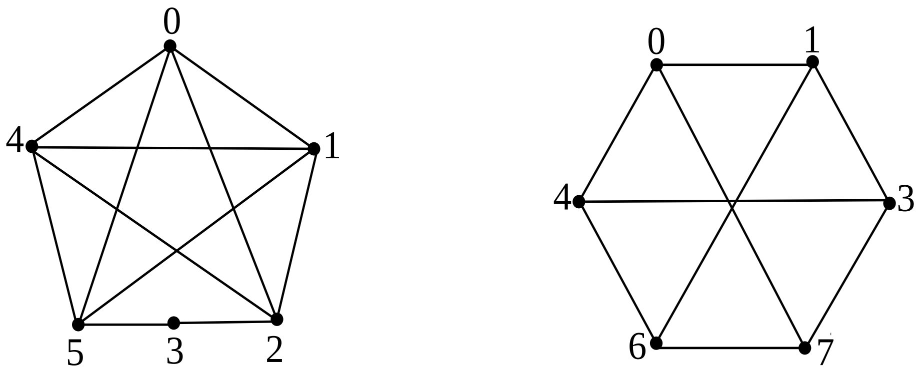

Figure 2: Subgraphs of G_7 and G_9, which are homeomorphic to K_5 and $K_{3,3}$, respectively.

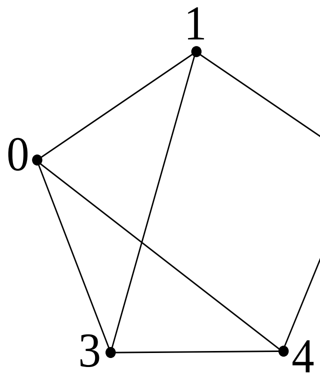

Figure 3: Subgraph of G_5, which is homeomorphic to K_4.

6 Conclusion

In this paper, we have introduced a new graph structure, called unitary addition Cayley graph. An important outcome of this paper is that for even values of n, unitary Cayley graphs and unitary addition Cayley graphs are isomorphic. Some other properties related to degrees of vertices, number of edges, girth, Eulerian and Hamiltonian cycles, partite classes, planarity and outerplanarity have also been examined, which we hope will set the tone for further research on this new graph structure.

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