NNTDM 17 (2011), 3, 18-21

On the Diophantine equation $y^n = f(x)^n + g(x)$

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Abstract: In this paper, we simplify the algorithm of Szalay [5].
Keywords: Diophantine equation, Irreducible polynomial, Height, Monic polynomial.
AMS Classification: 11B41

1 Introduction

Consider the equation

$$y^n = f(x)^n + g(x) \tag{1}$$

where n is a positive integer, $f(x)$ and $g(x)$ are non-zero rational polynomials, $f(x)$ has a positive leading coefficient and $deg(g(x)) < (n-1)deg(f(x))$. In 1887, Runge [3] proved: Let $f(x)$ be a polynomial of degree n with integral coefficients and let m be a positive integer. If $f(x) - y^m$ is irreducible over the rational field $\mathbb{Q}$, $\gcd(n, m) > 1$ and Runge's condition is satisfied (for Runge's condition, see [3]), then all integral solutions (x, y) of the equation $y^m = f(x)$ satisfy

$$|x| \leq d^{2n-d} \left(\frac{n}{d} + 2\right)^d (h+1)^{n+d},$$

where d is a divisor of $\gcd(m, n)$, $h = \max\{H(f(x)), 1\}$ and $H(f(x))$ is the height of $f(x)$. In 1969, Baker [1] found an upper bound for all integral solutions (x, y) of the equation $y^2 = f(x)$ in which $f(x)$ is a separable polynomial of degree $n \geq 5$ with integer coefficients. In 1999, Poulakis [2] described a method to solve equation 1, when the discriminant of the righthand of equation 1 is non-zero, $deg(f(x)) = n = 2$ and $g(x) \neq 0$. In 2000, Szalay [4] generalized the algorithm of Poulakis [2], when $g(x)$ is non-zero, n is even and the lefthand of equation 1 is monic. In 2002, Szalay [5] generalized the algorithm of Szalay [4].

Our aim in this paper is to present a simplest algorithm better than the algorithm of Szalay [5]. We plan this paper as follows. In section 2, we study the algorithm with examples. In section 3, we give a proof for correctness of the algorithm. Throughout of this paper, we use the following notations. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^-$ and $\mathbb{R}$ are natural, integer, negative integer and real sets, respectively, and $max\ A$ is the maximum number of the real subset A.

2 The Algorithm

In this algorithm, we consider equation (1) with a restriction that $f(x)$, $g(x) > 0$ for any positive integer x and we find an upper bound for all solutions $(x, y) \in \mathbb{N} \times \mathbb{Z}$.

Step 1. Find the least positive integer δ such that $\delta f(x)$ and $\delta^n g(x)$ have integer coefficients.

Case 1. $g(x)$ has a positive leading coefficient.

Step 2. Set

$$p(x) = \sum_{i=1}^{n} \binom{n}{i} (\delta f(x))^{n-i} - (\delta^n g(x)).$$

and

$$U = \{x \in \mathbb{R} : x > 0 \text{ and } p(x) = 0\}.$$

Step 3. If U is empty, then the given equation has no solution $(x, y) \in \mathbb{N} \times \mathbb{Z}$.

Step 4. All solutions $(x, y) \in \mathbb{N} \times \mathbb{Z}$ of the equation 1 satisfy

$$x \leq max\ U.$$

Case 2. $g(x)$ has a negative leading coefficient.

Step 2. Set

$$q(x) = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} (\delta f(x))^{n-i} + (\delta^n g(x)).$$

and

$$V = \{x \in \mathbb{R} : x > 0 \text{ and } q(x) = 0\}.$$

Step 3. If V is empty, then the given equation has no solutions $(x, y) \in \mathbb{N} \times \mathbb{Z}$.

Step 4. Each positive integral solution (x, y) of the equation 1 satisfies

$$x \leq max\ V.$$

Note.

(i) Consider equation (1). Replace x by $x' + \alpha$ in equation (1) where α is the least non-negative integer such that $f(x' + \alpha)$, $g(x' + \alpha) > 0$ for each positive integer x'. Now, we can use the above algorithm to find an upper bound for all solutions $(x', y) \in \mathbb{N} \times \mathbb{Z}$ of the equation $y^n = f(x' + \alpha)^n + g(x' + \alpha)$. From this, we can calculate the upper bound for all solutions $(x, y) \in \mathbb{N} \times \mathbb{Z}$, because $x = x' + \alpha$.

(ii) Also, we can use the same algorithm for finding a lower bound for x of all solutions $(x, y) \in \mathbb{Z}^- \times \mathbb{Z}$ by replacing of x by $-x$ in equation (1).

Example 1. $y^3 = x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 4x + 1$,
$f(x) = x^2 + x + 1$, $g(x) = x$, $\delta = 1$ *and* $n = 3$.
So $p(x) = 3x^4 + 6x^3 + 12x^2 + 8x + 7$.
Therefore, $U = \phi$*. Hence, the given equation has no solutions* $(x, y) \in \mathbb{N} \times \mathbb{Z}$*. Also, all solutions* $(x, y) \in \mathbb{Z}^- \times \mathbb{Z}$ *satisfy* $-1 \leq x$*. It is clear that* $(-1, 0)$ *is the only one solution of the given equation.*

In the following example, we have taken the same of Example 2 of Szalay [4].

Example 2. $y^2 = x^4 - 2x^3 + 2x^2 + 7x + 3$,
$f(x) = x^2 - x + \frac{1}{2}$, $g(x) = 8x + \frac{11}{4}$, $\delta = 2$ *and* $n = 2$.
So $p(x) = 4x^2 - 36x - 8$ *and*

$$max\ U = \frac{9 + \sqrt{90}}{2}.$$

Hence, by the above algorithm, all positive integral solutions (x, y) *of the given equation satisfy* $x \leq 9$. *Also, all solutions* $(x, y) \in \mathbb{Z}^- \times \mathbb{Z}$ *satisfy* $-7 \leq x$.

3 Proof of Trueness of the Algorithm

Case 1. $g(x)$ has a positive leading coefficient.

Claim 1. If $p(u) > 0$ for any positive integer u, then there does not exist any integer v such that (u, v) is a solution of equation (1).

Suppose there is an integral solution (x, y) for equation 1, such that $x > 0$ and $p(x) > 0$. Since $p(x) > 0$,

$$\sum_{i=1}^{n} \binom{n}{i} (\delta f(x))^{n-i} > (\delta^n g(x)).$$

Add $(\delta f(x))^n$ on both sides, then we get

$$(\delta f(x))^n + \sum_{i=1}^{n} \binom{n}{i} (\delta f(x))^{n-i} > (\delta f(x))^n + \delta^n g(x).$$

This implies that

$$(\delta f(x) + 1)^n > (\delta f(x))^n + \delta^n g(x) > (\delta f(x))^n,$$

since $x > 0$. Therefore,

$$(\delta f(x) + 1)^n > (\delta y)^n > (\delta f(x))^n.$$

This means that there is an integer between consecutive two integers. This is a contradiction. This proves Claim 1.

If U is empty, then the polynomial $p(x)$ has no real root $x > 0$. This means that $p(x)$ does not cross the x-axis in the positive side. Since the leading coefficient of $p(x)$ is positive, $p(x) > 0$ for each real number $x > 0$. Therefore, by Claim 1, we get the result.

Consider the case U is non-empty. Suppose there is a solution $(x, y) = (u, v) \in \mathbb{N} \times \mathbb{Z}$ for equation (1), such that $u > max\ U$. Then, $p(u) > 0$, since the leading coefficient of $p(x)$ is positive. So, by Claim 1, we get the contradiction. Hence, the above two subcases prove this case.

Case 2. $g(x)$ has a negative leading coefficient. This case follows the same methodology of Case 1.

From Case 1 and Case 2, we get the rightness of the algorithm.

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