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On the classes of Steiner loops of small orders

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Abstract. According to the number of sub-SL(8)s (sub-STS(7)s), there are five classes of sloops SL(16)s (STS(15)s) [2, 5].) In [4] the author has classified SL(20)s into 11 classes. Using computer technique in [10] the authors gave a large number for each class of SL(20)s. There are only simple SL(22)s and simple SL(26)s. So the next admissible cardinality is 28. Also, all SL(32)s are classified into 14 classes in [3]. We try to generalize the classification of SL(20)s given in [4] for SL(2n)s for each possible n and applying this method for n = 14 to classify all possible classes of SL(28)s. Consequently, we can establish all classes of nonsimple SL(28)s and all classes of semi-planar SL(28)s (STS(27)s). In this article, we show in section 3 that there are nine classes of SL(28)s (STS(27)s having one sub-SL(14) (sub-STS(13)) and r sub-SL(8)s (sub-STS(7)s) for r = 0, 1, 2, 3, 4, 5, 8, 11 or 16. All these sloops are subdirectly irreducible having exactly one proper homomorphic image isomorphic to SL(2). In section 4, we construct all classes of semi-planar SL(28)s. Such SL(28)s (STS(27)s) have r sub-SL(8)s (sub-STS(7)s) for r =1, 2, 3, 4, 5, 8, 11, 16 but no sub-SL(14) (sub-STS(13)).

In [4] is determined a necessary and sufficient condition for a sub-SL(2) = {1, x} of an SL(2n) to be normal. This result supplies us with the following two facts. First, there is another nonsimple subdirectly irreducible SL(2n) having exactly one proper homomorphic image isomorphic to an SL(n). Accordingly, we can construct all classes of nonsimple subdirectly irreducible SL(28)s. Second fact is that if an SL(2n) has a simple sub-SL(n) and (n- 1)(n - 2)/6 sub-SL(8)s passing through a non-unit element, then SL(2n) is isomorphic to the direct product SL(n) × SL(2). According to the result of section 3 and the above two facts, we may say that there are 8 simple classes of SL(28)s and only 11 classes of nonsimple SL(28)s, all these classes have no sub-SL(10)s. In the last section, we construct an example for each class given above of nonmsimple and simple (semi-planar) SL(28)s (STS(27)s). Finally, we review the classes of SL(2n)s (STS(2n -1)s) in 3 tables for 2n = 16, 20, and 28.

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1 Introduction

A Steiner loop (briefly sloop) is a groupoid $\boldsymbol{S} = (S; \cdot, 1)$ with neutral element 1 satisfying the identities:

$$x \cdot x = 1\,,\; x \cdot y = y \cdot x\,,\; x \cdot (x \cdot y) = y\,.$$

So, sloops are quasigroups [7, 12].

We use the abbreviations **SL**(n) for a sloop of cardinality n. A sloop is called Boolean if it satisfies the associative law $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, i.e., if the sloop is a Boolean group.

A Steiner triple system is a pair (P; B), where P is a set of points and B is a set of 3-element subsets of P called blocks such that for distinct points p_1, $p_2 \in P$, there is a unique block $b \in B$ with $\{p_1, p_2\} \subseteq b$. If the cardinality of the set of points P is equal to m, the Steiner triple system (P; B) will be denoted by **STS**(m). It is well known that the necessary and sufficient condition for the existence of an **STS**(m) is $m \equiv 1$ or 3 (mod 6) [11]. There is a one to one correspondence between sloops and Steiner triple systems given by the relation:

$$x \cdot y = z \Leftrightarrow \{x, y, z\} \text{ is a block [7, 11, 12].}$$

Quackenbush [12] proved that the congruences of sloops are permutable, regular, and Lagrangian. A subsloop $\boldsymbol{S}$ of a sloop $\boldsymbol{L}$ is called normal iff $(x \cdot y) \cdot S = x \cdot (y \cdot S)$ for all $x, y \in L$. Also in [12] Quackenbush proved that if $\boldsymbol{S}$ is a subsloop of $\boldsymbol{L}$ and $|L| = 2\,|S|$, then $\boldsymbol{S}$ is normal.

Distinct three points x, y, z form a triangle if $\{x, y, z\}$ does not form a block (or equivalently, if $\{x, y, z\}$ does not contain the identity element and $x \cdot y \neq z$). An **STS** is planar if it is generated by every triangle. A planar **STS**(m) exists for each $m \geq 7$ and $m \equiv 1$ or 3 (mod 6) [6]. The sloop associated with a planar triple system is also called planar. Quackenbush [12] showed that the only nonsimple finite planar sloop has 8 elements.

A semi-planar sloop is a simple sloop each of whose triangles generates either the whole sloop or a sub-**SL**(8). The **STS** associated with a semi-planar sloop is called a semi-planar **STS** or more precisely a semi-planar **STS** with sub-**STS**(7)s. The author [1] gave a construction of semi-planar sloops **SL**($2n$) for $n > 3$.

More about sloops can be found in [7, 12]. We will use in this article some basic concepts of universal algebra [8] and of graph theory [9].

There is a well-known classification of all **SL**(16)s into five classes based on the number of sub-**SL**(8)s [2]. All **SL**(16)s having no sub-**SL**(8)s are simple and all simple **SL**(16)s are planar. Except the Boolean **SL**(16) and the class of simple **SL**(16)s there are exactly three classes of nonsimple subdirectly irreducible **SL**(16)s [2]. All **SL**(22)s, and all **SL**(26)s are simple. In [4] the author has classified **SL**(20)*s* into 5 simple classes and 6 nonsimple classes. The next admissible orders for sloops is of cardinality 28. According to the algebraic and combinatory properties of each class of **SL**(28)s, we exhibit all classes of **SL**(28)s having no sub-**SL**(10). We describe how can one construct an example of each class of **SL**(28)s.

In sections 3 and 4, based on the cardinality and the number of the (normal) subsloops, we will exhibit all possible classes **SL**(28)s (all classes of **SL**(28)s having no sub-**SL**(10)). Note that a sub-**SL**(n) is always normal in an **SL**($2n$) [12]. Similarly, as for n = 10 in [4] and for n = 14, we can generalize this classification, if we begin with a planar **SL**(n) = $\boldsymbol{L_1}$ having a 1-factorization $\boldsymbol{F}(\boldsymbol{L_1})$ on $\boldsymbol{L_1}$ satisfying that the sub-1-factorizations of $\boldsymbol{F}(\boldsymbol{L_1})$ exist only on the sub-**SL**(4)s of $\boldsymbol{L_1}$. Then we may construct all possible classes of nonsimple subdirectly irreducible **SL**($2n$) for each possible n.

We review the results of this article in the following:

1. In section 3, there are 10 classes of nonsimple **SL**(28)s having exactly one sub-**SL**(14) and r sub-**SL**(8)s for r = 0, 1, 2, 3, 4, 5, 8, 11, 16, 26. For n = 14 and r = 26, the doubling constructed **SL**($2n$) = $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ with the planar sub-**SL**(n) = $\boldsymbol{L_1}$ is isomorphic to the direct product **SL**(n) × **SL**(2). All **SL**(28)s except for r = 26 are nonsimple subdirectly irreducible. In general, if the number r of sub-**SL**(8)s = (n - 1)(n - 2)/6 for $n \geq 10$, then the doubling constructed **SL**($2n$) = $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ with the planar sub-**SL**(n) = $\boldsymbol{L_1}$ is isomorphic to the direct product $\boldsymbol{L_1}$ × **SL**(2). The number r depends on n, so the values of r are determined individually for each value of n.
2. Due to Doyen [6], there are planar **STS**(n - 1)s for each possible $n \geq 10$. The associated planar **SL**(n)s are simple for each possible $n > 10$ [12]. Indeed, planar **SL**(n)s have no nontrivial subsloops.

   In section 4, we will show that there are eight classes of semi-planar **SL**(28)s with r sub-**SL**(8)s with r = 1, 2, 3, 4, 5, 8, 11, 16. All of these semi-planar **SL**(28)s are simple but not planar. In addition, the associated **STS**(27)s are semi-planar (each triangle generates a sub-**STS**(7) or the whole **STS**(27)).
3. According to the construction given in section 4, there is a class of **SL**($2n$)s having (n - 1)(n - 2)/6 sub-**SL**(8)s and no sub-**SL**(n). Theorem 6 tells us that these sloops are subdirectly irreducible other than the constructed **SL**($2n$) = $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ given in section 3 and they have exactly one proper homomorphic image isomorphic to **SL**(n) (not Boolean)), for each possible number $n \geq 10$. So, we may say that all classes of nonsimple subdirectly irreducible **SL**(28)s are determined.
4. In section 5 and 6, we describe how can one construct an example for each class of **SL**(28). Also we review all classes of the cardinalities 16 and 20, and all classes of cardinality 28 having no sub-SL(10)s in three tables.

2 Construction of an $\mathbf{SL}(2n) = 2\otimes_\alpha \boldsymbol{L_1}$

Using the doubling construction $\mathbf{SL}(2n)$s [11], we will exhibit in this section some properties of subsloops of $\mathbf{SL}(2n)$s .

Let $\boldsymbol{T}_1 = (P^*_1; B_1)$ be an $\mathbf{STS}(n\text{-}1)$ and its corresponding sloop $\boldsymbol{L_1} = (P_1; \cdot , e)$, where $P^*_1 = \{a_1, \dots , a_{n\text{-}1}\}$ and $P_1 = P^*_1 \cup \{e\}$. Consider the set of 1-factors defined by $F_i = \{e\, a_i \} \cup \{a_l\ a_k : a_l \,.\, a_k = a_i$ and $a_i, a_l, a_k \in P_1\}$, then the class $\boldsymbol{F} = \{F_1, F_2, \dots , F_{n\text{-}1}\}$ forms a 1-factorization of the complete graph $\mathbf{K}_n$ on the set of vertices P_1. The 1-factorization $\boldsymbol{F}$ will be denoted by $\boldsymbol{F}(\boldsymbol{L_1})$.

By taking the set $P_2 = \{b, b_1, b_2, \dots , b_{n\text{-}1}\}$ with $P_1 \cap P_2 = \varnothing$ and

$$G_i = \{b\ b_i \} \cup \{b_l\ b_k : a_l \cdot a_k = a_i \text{ for } i \notin \{ l, k\}\},$$

then the class of 1-factors $\boldsymbol{G} = \{G_1, G_2, \dots , G_{n\text{-}1}\}$ forms a 1-factorization of the complete graph $\mathbf{K}_n$ on the set of vertices P_2 and it will be denoted by $\boldsymbol{G}(P_2)$. We consider the doubling construction of triple systems [11] as follows: $\mathbf{STS}(2n\ \text{-}1) = (P^*; B)$, where $P^* = P^*_1 \cup P_2$ and the set of triples $B = B_1 \cup B_{12}$, where $B_{12} = \{\{a_i, b_j, b_k\} :\ b_j\ b_k \in G_{\alpha(i)}\}$, for any permutation α on the set $N = \{1, \dots , n - 1\}$. The constructed $\mathbf{STS}(2n\ \text{-}1) = (P^*; B)$ and the associated sloop $\mathbf{SL}(2n) = (P; \cdot , e)$ will be denoted by $2\otimes_\alpha \boldsymbol{T_1}$ and $2\otimes_\alpha \boldsymbol{L_1}$, respectively.

We consider the following notation in the whole article, the $\mathbf{STS}(n\ \text{-}1) = (N; X)$, where X is defined by: $\{i, j, k\} \in X$ if and only if $\{a_i, a_j, a_k\} \in B_1$; i.e. $(N; X)$ is an $\mathbf{STS}(n - 1)$ isomorphic to $(P^*_1; B_1)$.

We observe that $\boldsymbol{L_1}$ is a normal subsloop of $2\otimes_\alpha \boldsymbol{L_1}$ for any permutation α. If we choose α = the identity, then the constructed sloop $\boldsymbol{L} = 2\otimes_\alpha \boldsymbol{L_1}$ is isomorphic to the direct product of $\mathbf{SL}(n) = \boldsymbol{L_1}$ and the 2-element sloop $\mathbf{SL}(2)$.

The following lemma clarifies the relation between the subsloops of the constructed sloop $2\otimes_\alpha \boldsymbol{L_1}$ and the normal subsloop $\boldsymbol{L_1}$.

Lemma 1 [4]. *Let* $\mathbf{L} = (P = P_1 \cup P_2; \cdot , e)$ *be a sloop of cardinality* $2n$ *with the subsloop* $\boldsymbol{L_1} = (P_1; \cdot , e)$ *of cardinality n. Then any subsloop* $\mathbf{S}$ *of* $\mathbf{L}$ *with* $S - P_1 \neq \varnothing$ *satisfies* $|S \cap P_1| = (1/2)\, |S|$.

Accordingly, if $\boldsymbol{L_1}$ is a planar sloop, then $|S \cap P_l| = (1/2)\,|S| = 1$, 2 or 4. Also, the subsloops of the constructed $\mathbf{SL}(2n) = 2\otimes_\alpha \boldsymbol{L_1}$ satisfies the above lemma. This means that if $\boldsymbol{L_1}$ is a planar sloop, then the constructed $\mathbf{SL}(2n) = 2\otimes_\alpha \boldsymbol{L_1}$ has only subsloops of cardinality 2, 4, or 8.

In this article, we consider a planar sloop $\boldsymbol{L_1}$. So the only possible nontrivial subsloops other than $\boldsymbol{L_1}$ of $\mathbf{2}\otimes_\alpha\boldsymbol{L_1}$ are of cardinality 8. Accordingly, we need only identify all sub-1-factorizations of $\mathbf{K_4}$ of the associated 1-factorization $\boldsymbol{F}(\boldsymbol{L_1})$ and also of $\boldsymbol{G}(P_2)$ to determine all sub-**SL**(8)s of $\mathbf{2}\otimes_\alpha\boldsymbol{L_1}$.

For each block $\{a_i, a_j, a_k\} \in B_1$, there is a sub-1-factorization $f = \{f_i = \{e\ a_i, a_j\ a_k\}, f_j = \{e\ a_j, a_i\ a_k\}, f_k = \{e\ a_k, a_i\ a_j\}\}$ of $\boldsymbol{F}(\boldsymbol{L_1})$. Conversely, if there is a sub-1-factorization of $\boldsymbol{F}(\boldsymbol{L_1})$ on the 4-element subset $\{e, a_i, a_j, a_k\}$, then $\{a_i, a_j, a_k\}$ is a block in B_1. This means that there is a one-one correspondence between the set of blocks of B_1 and the sub-1-factorizations of $\mathbf{K}_4$ in $\boldsymbol{F}(\boldsymbol{L_1})$.

The next result is valid for n = 10 as in [4], we generalize it as follows:

Lemma 2. *Let* $\boldsymbol{T}_1 = (P^*_1; B_1)$ *be a planar* **STS**(n-1) *and* $\boldsymbol{L_1} = (P_1;\ \cdot\ , e)$ *be the corresponding planar sloop for* $n > 8$. *If the associated 1-factorization* $\mathbf{F}(\boldsymbol{L_1})$ *for a planar* $\boldsymbol{L_1}$ *has sub-1-factorizations of* $\mathbf{K}_4$ *only on the 4-element sub-*$\mathbf{SL}$(4)*s, then the associated 1-factorization* $\boldsymbol{F}(\boldsymbol{L_1})$ *has exactly* (n-1)(n-2)/6 (*the number of blocks of* B_1) *sub-1-factorizations of* $\mathbf{K}_4$. *Especially, for n* = 10 *or* 14, *the 4-element subsets of the sub-1-factorizations of* $\mathbf{K}_4$ *are exactly the sub-*$\mathbf{SL}$(4)s = $\{e, a_i, a_j, a_k\}$ *of* $\boldsymbol{L_1}$.

Proof. Clearly, there is always a sub-1-factorization in the associated 1-factorization $\boldsymbol{F}(\boldsymbol{L_1})$ on each 4-element sub-**SL**(4).

Let $\{e, a_i, a_j, a_k\}$ be a subsloop of $\boldsymbol{L}_1$, then the sub-1- factors $f_i = \{e\ a_i, a_j\ a_k \subseteq F_i, f_j = \{e\ a_j, a_i\ a_k\} \subseteq F_j$ and $f_k = \{e\ a_k, a_i\ a_j\} \subseteq F_k$ form a sub-1-factorization of $\boldsymbol{F}(\boldsymbol{L_1})$. Since $\boldsymbol{L_1}$ is planar, we can say that there is no another sub-1-factorization in the set of factors $\{F_i, F_j, F_k\}$. Otherwise, the sub-1-factors $f_i^` = \{e\ a_i, a_j\ a_k, x\ y, z\ w\} \subseteq F_i, f_j^` = \{e\ a_j, a_i\ a_k, x\ z, y\ w\} \subseteq F_j$ and $f_k^` = \{e\ a_k, a_i\ a_j, x\ w, y\ z\} \subseteq F_k$ forms a sub-**SL**(8) of the planar sloop $\boldsymbol{L_1}$ on the set $\{e, a_i, a_j, a_k, x, y, z, w\}$, which is impossible.

Assume that $\{e, a_i, a_j, a\}$ does not form a sub-**SL**(4) of $\boldsymbol{L_1}$, we will prove that the set of factors $\{F_i, F_j, F_a\}$ has no sub-1- factorizations on the subset $\{e, a_i, a_j, a\}$.

For *n* = 10: The author [4] has proved that the sub-1-factorizations in $\boldsymbol{F}(\boldsymbol{L_1})$ occurs only on the 4-element sub-**SL**(4)s of $\boldsymbol{L_1}$.

Since the number of blocks of B_1 is 12, the 1-factorizations $\boldsymbol{F}(\boldsymbol{L_1})$ has exactly 12 sub-1-facorizations of $\mathbf{K}_4$. Each of them is defined on a sub-**SL**(4).

For *n* = 14: The order r of the complete graph $\mathbf{K}_r$ for a sub-1-factorization of $\boldsymbol{F}(\boldsymbol{L_1})$ is equal to 4 or 6. Indeed, a sub-1-factorization of $\mathbf{K_6}$ does not form a subsloop of $\boldsymbol{L_1}$. Consequently, we are focusing only with the sub-1-factorizations of $\mathbf{K_4}$. We need only consider the case when $\{e, a_i, a_j, a\}$ does not form a subsloop of $\boldsymbol{L_1}$; hence we may assume $a_i \cdot a_j = a_k \neq a$.

If there is a sub-1-factorization on a 4-element subset $\{x, y, z, w\}$, then $e \in \{x, y, z, w\}$. Otherwise, assume $e \notin \{x, y, z, w\}$, and $\{x\ y, z\ w\} \subseteq F_i$, $\{x\ z, y\ w\} \subseteq F_j$, and $\{x\ w, y\ z\} \subseteq F_a$ form

a sub-1-factorization of $\mathbf{K}_4$. But we know that $e\,a_i \in F_i$, $e\,a_j \in F_j$ and $e\,a_k \in F_k$, then $\{e, a_i, a_j, a\} \cap \{x, y, z, w\} = \emptyset$. Without loss of generality, we may choose the following set of 1-factors: $F_i = \{e\,a_i, a_j\,a_k, x\,y, z\,w, a\,b, c\,d, f\,g\}$, $F_j = \{e\,a_j, a_i\,a_k, x\,z, y\,w, a\,c, b\,g, d\,f\}$, $F_a = \{e\,a, a_i\,b, a_j\,c, x\,w, y\,z, a_k\,f, d\,g\}$, and $F_k = \{e\,a_k, a_i\,a_j, a\,f, x\,b, y\,c, z\,d, w\,g\}$. We note that the points x, y, z, w must lie in four different edges in the 1-factor F_k. This means that $F_x = \{e\,x, a_i\,y, a_j\,z, a\,w, a_k\,b, c\,f, d\,g\}$, which contradicts the fact that $F_x \cap F_a = \emptyset$. Notice that any other choice leads to the same contradiction. This means that the set of factors $\{F_i, F_j, F_a\}$ has only sub-1- factorization of $\mathbf{K_4}$, if $\{e, a_i, a_j, a\}$ forms a sub-**SL**(4).
Since the number of blocks of B_1 in **STS**(13) is 26, the 1-factorizations $\boldsymbol{F}(\boldsymbol{L_1})$ has exactly 26 sub-1-facorizations of $\mathbf{K}_4$, each of them is defined on a sub-**SL**(4).

In general, **for $n \geq 16$**, if the associated 1-factorization $\boldsymbol{F}(\boldsymbol{L_1})$ for a planar $\boldsymbol{L_1}$ has sub-1-factorizations only on the 4-element sub-**SL**(4)s = $\{e, x, y, z\}$, then the associated 1-factorization $\boldsymbol{F}(\boldsymbol{L_1})$ has exactly $(n\text{-}1)(n\text{-}2)/6$ sub-1-factorizations of $\mathbf{K}_4$. This completes the proof. □

In general, the associated 1-factorization $\boldsymbol{F}(\boldsymbol{L_1})$ may has a sub-1-factorization on a 4-element subset $\{x, y, z, w\}$ of P_1 with $e \notin \{x, y, z, w\}$, even if the **SL**(n) is planar. For example for $n = 16$, there are planar and not planar **SL**(16)s [4]. Also, there is a planar **SL**(16) = $\boldsymbol{L_1}$ in which the associated 1-factorization $\boldsymbol{F}(\boldsymbol{L_1})$ has a sub-1-factorization on a 4-element subset satisfying $e \notin \{x, y, z, w\}$. Moreover, there is a planar **SL**(16) = $\boldsymbol{L_1}$ in which the associated 1-factorization $\boldsymbol{F}(\boldsymbol{L_1})$ has only sub-1-factorizations on the 4-element sub-**SL**(4)s. So we restrict our discussion in this article for planar **SL**(n) = $\boldsymbol{L_1}$ in which the associated 1-factorization $\boldsymbol{F}(\boldsymbol{L_1})$ has only sub-1-factorizations on the 4-element sub-**SL**(4)s.

Also, we note that the 1-factorization $\boldsymbol{G}(P_2)$ has only sub-1-factorizations of $\mathbf{K_4}$ on the 4-element subset $\{b, b_i, b_j, b_k\}$ of P_2, if and only if $\{e, a_i, a_j, a_k\}$ is a sub-**SL**(4) of $\boldsymbol{L_1}$. Moreover, the sub-1-facorizations of $\mathbf{K}_4$ in both $\boldsymbol{F}(\boldsymbol{L_1})$ and $\boldsymbol{G}(P_2)$ are determined by:

$\boldsymbol{f} = \{f_i = \{e\,a_i, a_j\,a_k\}, f_j = \{e\,a_j, a_i\,a_k\}, f_k = \{e\,a_k, a_i\,a_j\}\}$ and
$\boldsymbol{g} = \{g_i = \{b\,b_i, b_j\,b_k\}, g_j = \{b\,b_j, b_i\,b_k\}, g_k = \{b\,b_k, b_i\,b_j\}\}$ for all $\{i, j, k\} \in X$.

Accordingly, we may easily verify the following lemma.

Lemma 3 [4]. *Let* $C_1 = \{e, a_i, a_j, a_k\}$ *and* $C_2 = \{b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ *be 4-element subsets of* P_1 *and* P_2 *, respectively. Then* $\mathbf{2}\otimes_\alpha \boldsymbol{C_1} = (C_1 \cup C_2; \cdot, e)$ *is a sub-***SL**(8) *of the construction* $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ *if and only if* $\{i, j, k\}$ *and* $\{\alpha(i), \alpha(j), \alpha(k)\}$ *are lines in X.*

Accordingly, we may say that the only possible nontrivial subsloops of $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ are $\boldsymbol{L_1}$ (exactly one sub-**SL**(n)) and r sub-**SL**(8)s, the values of r depends also on n and satisfies $0 \leq r \leq$

(n -1) (n -2)/6. Each of sub-**SL**(8) intersects $\boldsymbol{L_1}$ in a sub-**SL**(4). This implies that any proper subsloop $\boldsymbol{S}$ of $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ with $|S| > 4$ and $\boldsymbol{S} \neq \boldsymbol{L_1}$ is determined by the 8-element subset $S = \{e, a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ such that $\{i, j, k\}$ and $\{\alpha(i), \alpha(j), \alpha(k)\} \in X$.

3 Subdirectly irreducible sloops of cardinality 2*n*

Any sloop of cardinality $2n$ with a planar sub-**SL**(n) has no more subsloops of cardinality n and has at most (n-1)(n-2)/6 subsloops of cardinality 8. In particular, the direct product of a planar sloop **SL**(n) and the **SL**(2) has exactly one sub-**SL**(n) and (n-1)(n-2)/6 sub-**SL**(8)s. On the other hand, planar **SL**($2n$)s have no nontrivial subsloops.

A nonsimple subdirectly irreducible **SL**($2n$) has exactly one normal subsloop ≅ **SL**(n) or **SL**(2). In the next theorem we exhibit all nonsimple subdirectly irreducible **SL**($2n$)s having a planar sub-**SL**(n) for each possible n. Note that we restrict our discussion in this article for planar **SL**(n) = $\boldsymbol{L_1}$ in which the associated 1-factorization $\boldsymbol{F}(\boldsymbol{L_1})$ has only sub-1-factorizations on the 4-element sub- **SL**(4)s.

Theorem 4. *The constructed sloop* $\mathbf{2}\otimes_\alpha \boldsymbol{L_1} = (P = P_1 \cup P_2; \cdot , e)$ *with a planar subsloop* **SL**(n) = $\boldsymbol{L_1}$ *is isomorphic to the direct product of the subsloop* $\boldsymbol{L_1}$ *and the* 2*-element sloop* **SL**(2), *if and only if* $2 \otimes_\alpha \boldsymbol{L_1}$ *has* (n-1)(n-2)/6 *sub*-**SL**(8)s, *otherwise* $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ *is nonsimple subdirectly irreducible. Moreover, the constructed sloop* $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ *has exactly r subsloops of cardinality* 8 *if and only if the permutation* α *transfers r lines into r lines of X for certain numbers r depending on n and satisfying* $0 \leq r \leq (n-1)(n-2)/6$, *where* $(N, X) \cong (P_1^*, B_1)$ *and* $N = \{1, 2, \ldots, n - 1\}$.

Proof. Let $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ have (n-1)(n-2)/6 sub-**SL**(8)s, this means that $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ has maximal numbers of sub-**SL**(8)s. According to Lemmas 1, 2 and 3, the image of each line in X is again a line in X, then $\alpha(X)$ ′ $\{\{\alpha(i), \alpha(j), \alpha(k)\} : \text{for all } \{i, j, k\} \in X\} = X$. Consider the map φ from $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ to the direct product $\boldsymbol{L_1} \times \{0, 1\}$ by $\varphi(e) = (1, 0)$, $\varphi(b) = (1, 1)$, $\varphi(a_i) = (a_i, 0)$ and $\varphi(b_i) = (a_{\alpha^{-1}(i)}, 1)$. It is a routine matter to proof that φ is an isomorphism. Notice that $\varphi(a_i\, b_j) = \varphi(b_k)$ if $b_j\, b_k \in G_{\alpha(i)}$. This means that $\{\alpha(i), j, k\}$ is a block in X. Also, $\varphi(b_k) = (a_{\alpha^{-1}(k)}, 1)$ and $\varphi(a_i)\, \varphi(b_j) = (a_i, 0)\, (a_{\alpha^{-1}(j)}, 1) = (a_i\, a_{\alpha^{-1}(j)}, 1)$, but $\alpha^{-1}\{\alpha(i), j, k\} = \{i, \alpha^{-1}(j), \alpha^{-1}(k)\}$ is also a blook in X, so $\varphi(a_i)\, \varphi(b_j) = \varphi(b_k)$.

The sloop $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ has a normal sub-**SL**(n) = $\boldsymbol{L_1}$; another possible normal subsloop is the 2-element subsloop $\boldsymbol{C}_2$. In this case $\boldsymbol{C}_2$ must satisfy that $\boldsymbol{C}_2 \cap \boldsymbol{L_1} = \{e\}$; this means that the sloop $\mathbf{2}\otimes_\alpha \boldsymbol{L_1}$ is isomorphic to the direct product **SL**(n) × **SL**(2) and has exactly (n-1)(n-2)/6)) sub-**SL**(8)s. Therefore, if $\mathbf{2} \otimes_\alpha \boldsymbol{L_1}$ has r sub-**SL**(8)s with $r < (n-1)(n-2)/6$, then the congruence lattice

of $\mathbf{2} \otimes_\alpha \boldsymbol{L_1}$ has only one normal subsloop that is $\boldsymbol{L_1}$. Therefore, $\mathbf{2}\otimes_\alpha\boldsymbol{L_1}$ is subdirecly irreducible for all possible $r < (n\text{-}1)(n\text{-}2)/6$.
Let α transfer the line $\{i, j, k\} \in X$ into the line $\{\alpha(i), \alpha(j), \alpha(k)\} \in X$. According to Lemmas 3, we may directly say that $S = \{e, a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ forms a subsloop. Since α is a permutation on the set of points $N = \{1, 2, \ldots, n - 1\}$ of the system (N, X), it follows that the possible values of the number of lines r of X transferred into lines satisfy $0 \leq r \leq (n\text{-}1)(n\text{-}2)/6$. This completes the proof. □

The classification of **SL**($2n$)s depends on the number of sub-**SL**(8)s r and the possible values of r depends also on n. For this reason, we discuss the following result only for $n = 14$. For $n = 10$, X is the set of lines of the affine plane over **GF**(3), then the possible values of r are 0, 1, 2, 3, 4, 6 or 12 [4]. As we will see in section 5 that the possible values of r are 0, 1, 2, 3, 4, 5, 8, 11, 16 or 26 for $n = 14$.

Moreover, there is another class of subdirectly irreducible **SL**($2n$)s having exactly one proper normal sub-**SL**(2). It will be described as in the following theorem.

For each non-unit element x of a sloop the 2-element set $\{e, x\}$ forms always a subsloop, the next theorem supplies us with a necessary and sufficient condition for the sub-**SL**(2) = $\{e, x\}$ to be normal for a finite sloop. This helps us to determine the other classes of subdirectly irreducible **SL**($2n$)s.

Theorem 5 [4]. *Let* **L** *be a sloop of cardinality* $2\,n$. *A subsloop* $\mathbf{S} = \{e, x\}$ *is normal if and only if* **L** *contains* $(n - 1)(2n - 4)/12$ *sub-***SL**(8)*s including the element* x.

According to Theorem 4, if an **SL**($2n$) with a planar sub-**SL**(n) has $(n\text{-}\ 1)(n - 2)/6$ sub-**SL**(8)s, then **SL**($2n$) ≅ **SL**(n) × **SL**(2), which implies that **SL**($2n$) has a normal sub-**SL**(2) = $\{e, x\}$. Hence the factor sloop of **SL**($2n$) by the normal sub-**SL**(2) is isomorphic to **SL**(n). Each sub-**SL**(4) of the factor **SL**(n) forms a sub-**SL**(8) of **SL**($2n$) passing through x. This means that if an **SL**($2n$) has $(n\text{-}\ 1)(n - 2)/6$ sub-**SL**(8)s, then each sub-**SL**(8) passes through the sub-**SL**(2) = $\{e, x\}$. In addition, if the **SL**($2n$) has no sub-**SL**(n), then **SL**($2n$) is subdirectly irreducible other than the subdirectly irreducible **SL**($2n$) = $\mathbf{2} \otimes_\alpha \boldsymbol{L_1}$ given by Theorem 4. Consequently, we may say that all nonsimple subdirectly irreducible **SL**(28)s are determined.

In particular, Theorem 4 and 5 show that if the constructed **SL**($2n$) = $\mathbf{2}\otimes_\alpha\boldsymbol{L_1}$ has a simple planar sub-**SL**(n) = $\boldsymbol{L_1}$ and $(n - 1)(n - 2)/6$ sub-**SL**(8)s, then each sub-**SL**(8) passes through a normal sub-**SL**(2) = $\{e, x\}$. According to Lemma 1 and the definition of the constructed $\mathbf{2}\otimes_\alpha\boldsymbol{L_1}$, then $x = b$, hence the subsloops $\boldsymbol{L_1}$ and $\{e, b\}$ are normal in $\mathbf{2}\otimes_\alpha\boldsymbol{L_1}$. Therefore, $\mathbf{2}\otimes_\alpha\boldsymbol{L_1}$ is isomorphic to the direct product $\boldsymbol{L_1} \times \{e, b\}$.

4 Semi-planar sloops of cardinality 2*n*

We restrict our discussion in this section for **SL**(2*n*) for $n = 14$. A semi-planar sloop is a simple sloop each of whose triangles generates either the whole sloop or a sub-**SL**(8) " cf. [1]". The **STS**s associated with the semi-planar sloops will also be called semi-planar (or more precisely semi-planar with sub-**STS**(7)s). We note that all simple **SL**(16)s are planar; i.e. there is no semi-planar **SL**(16), which is not planar.

For $n > 16$, we discuss the classes of semi-planar **SL**(2*n*)s, individually. According to the value of *n* we determine the values of *r* (the number of sub-**SL**(8)s). For $n = 10$, we determined all classes of semi-planar **SL**(20) in [4]. Based on the number *r* of sub-**SL**(8)s of **SL**(28), we will determine all possible classes of semi-planar **SL**(28)s. So we will see that there are eight distinct classes of semi-planar **SL**(28)s.

In the following we will modify the above construction of the subdirectly irreducible **SL**(2*n*) = $\mathbf{2}\otimes_\alpha \boldsymbol{L}_1 = (P = P_1 \cup P_2; \cdot, e)$ to get a construction of a semi-planar sloop denoted by $\underline{\mathbf{2}}\otimes_\alpha \underline{\boldsymbol{L}}_1$.

Consider a constructed subdirectly irreducible **SL**(2*n*) = $\mathbf{2}\otimes_\alpha \boldsymbol{L}_1$ having a sub-**SL**(8), then the set of elements of sub-**SL**(8) can be considered $A = \{e, a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$. Also consider the **STS**(2*n* - 1) associated with the constructed subdirectly irreducible **SL**(2*n*) = $\mathbf{2}\otimes_\alpha \boldsymbol{L}_1$,; i.e., the associated **STS**(2*n* - 1) has a sub-**STS**(7) on the subset $A^* = \{a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$, where. $\{i, j, k\}$ and $\{\alpha(i), \alpha(j), \alpha(k)\}$ are lines in *X*.
By interchanging the following set of blocks. :

$$H = \{\{a_i, a_j, a_k\}, \{a_i, b_{\alpha(j)}, b_{\alpha(k)}\}, \{a_j, b_{\alpha(i)}, b_{\alpha(k)}\}, \{a_k, b_{\alpha(i)}, b_{\alpha(j)}\}\}$$

with the set of triples

$$R = \{\{b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}, \{b_{\alpha(i)}, a_j, a_k\}, \{b_{\alpha(j)}, a_i, a_k\}, \{b_{\alpha(k)}, a_i, a_j\}\},$$

we get again an **STS**(2*n* - 1) = $(P^* = P^*_1 \cup P_2; B - H \cup R)$. The constructed **STS**(2*n* - 1) and the associated **SL**(2*n*) will be denoted by $\underline{\mathbf{2}}\otimes_\alpha \underline{\boldsymbol{T}}_1$ and $\underline{\mathbf{2}}\otimes_\alpha \underline{\boldsymbol{L}}_1$, respectively, where $\underline{\mathbf{2}}\otimes_\alpha \underline{\boldsymbol{L}}_1 = (P = P_1 \cup P_2; \underline{\cdot}, e)$. Notice that the difference between the binary operations " $\underline{\cdot}$ " and " $\cdot$ " is only restricted on the subset of elements of A^*; i. e., $x \,\underline{\cdot}\, y = x \cdot y$ for all $x, y \in P - A^*$.

In the next lemma we will show that the new construction $\underline{\mathbf{2}}\otimes_\alpha \underline{\boldsymbol{L}}_1$ is a semi-planar sloop such that α transfers at least one line into a line and at most ((*n*-1)(*n*-2)/6) - (*n*-4) lines into ((*n*-1)(*n*-2)/6) - (*n*-4) lines of the **STS**(*n* - 1) = (*N*; *X*) for both $n = 10$ and 14. This result is proved for $n = 10$ in [4], in the following theorem we prove similar result for $n = 14$.

Theorem 6. *Let* $\boldsymbol{L}_1$ *be a sloop of cardinality* 14. *The constructed sloop* $\underline{\mathbf{2}}\otimes_\alpha \underline{\boldsymbol{L}}_1 = (P = P_1 \cup P_2; \underline{\cdot}, e)$ *has no sub*-**SL**(*n`*)*s for each possible n` satisfying* $8 < n` \leq n = 14$. *Also,* $\underline{\mathbf{2}}\otimes_\alpha \underline{\boldsymbol{L}}_1$ *is a semi-*

planar sloop having r sub-$\mathbf{SL}(8)$*s for each possible r, where r is the number of lines of the triple system* $(N; X)$ *transferred into r lines satisfying* $1 \leq r \leq ((n\text{-}1)(n\text{-}2)/6)\text{-}(n\text{-}4)$.

Proof. We use some times the letter n instead 14 to point out that this proof is valid for other values of n for example for $n = 10$ as in [4]. Let $S = \{x, y, z\}$ be a triangle in $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$. At first, we want to prove that the subsloop $\underline{<S>}$ generated by S in $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$ is equal to the whole sloop $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$ or a sub-$\mathbf{SL}(8)$.

Assume that $|\underline{<S>} \cap A| \leq 2$, where $A = \{e, a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$, then the subsloop $<S>$ in the sloop $\mathbf{2}\otimes_\alpha\boldsymbol{L}_1$ is the same subsloop $\underline{<S>}$ in $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$. The unique sub-$\mathbf{SL}(n)$ in $\mathbf{2}\otimes_\alpha\boldsymbol{L}_1$ is $\boldsymbol{L}_1$, hence if the subsloop $\underline{<S>}$ is a sub-$\mathbf{SL}(n)$, then $\underline{<S>} = <S> = \boldsymbol{L}_1$ contradicting the fact that $a_i \underline{\cdot} a_j = b_{\alpha(k)}$ in $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$.

Also, if $\underline{<S>} \cap A = \{e, b, a_i, b_{\alpha(i)},\}, \{e, b, a_j, b_{\alpha(j)}\}$ or $\{e, b, a_k, b_{\alpha(k)}\}$, then the subsloop $<S>$ of $\mathbf{2}\otimes_\alpha\boldsymbol{L}_1$ is the same subsloop $\underline{<S>}$ in $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$. For the same reason above, if $<S>$ is a sub-$\mathbf{SL}(n)$, then $\underline{<S>} = <S> = \boldsymbol{L}_1$ contradicting the fact that $b \in \underline{<S>}$. Moreover if $|\underline{<S>} \cap A| > 4$, then $\underline{<S>} = A$ or $\underline{<S>} \supset A$; i.e., $\underline{<S>}$ is a sub-$\mathbf{SL}(8)$ or $\underline{<S>} = \underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$.

Now, assume that $\underline{<S>} \cap A = \{e, b_{\alpha(i)}, a_j, a_k\}, \{e, b_{\alpha(j)}, a_i, a_k\}, \{e, b_{\alpha(k)}, a_i, a_j\}$ or $\{e, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ and $|\underline{<S>}| = n` \leq n$; i. e. $n` = 8$, 10 or 14 for $n = 14$. We will find a contradiction if $n`= 10$ or 14 for $n = 14$.

Each of these four possible blocks contains at least one element $b_{\alpha(t)}$ lying in P_2, for $t = i, j$ or k. By taking $b_{\alpha(t)} \in \underline{<S>} \cap A$, if b_r or $a_s \in \underline{<S>} - A$, then $b_{\alpha(t)} \underline{\cdot} b_r \in P_1$ or $b_{\alpha(t)} \underline{\cdot} a_s \in P_2$, This means that the $(n` - 4)$-element subset $\underline{<S>} - A$ consists of two disjoint $m$-element subsets $\{a_{s_1}, a_{s_2}, \dots, a_{s_m}\} \subseteq P_1$ and $\{b_{r_1}, b_{r_2}, \dots, b_{r_m}\} \subseteq P_2$ with $m = (n` - 4)/2$; i. e. $m = 3$ or 5 for $n = 14$.

For the first three cases: $\underline{<S>} \cap A = \{e, b_{\alpha(i)}, a_j, a_k\}, \{e, b_{\alpha(j)}, a_i, a_k\}$ or $\{e, b_{\alpha(k)}, a_i, a_j\}$, we have for

$a_t \in \underline{<S>} \cap A$ that $a_t \underline{\cdot} \{a_{s_1}, a_{s_2}, \dots, a_{s_m}\} \cap \{e, a_i, a_j, a_k\} = \varnothing$ and $a_t \underline{\cdot} \{a_{s_1}, a_{s_2}, \dots, a_{s_m}\} = a_t \cdot \{a_{s_1}, a_{s_2}, \dots, a_{s_m}\} \subseteq P_1$. Also, $a_t \underline{\cdot} \{a_{s_1}, a_{s_2}, \dots, a_{s_m}\} \neq \{a_{s_1}, a_{s_2}, \dots, a_{s_m}\}$ because $m = 3$ or 5 (not even), hence $\underline{<S>} - A$ contains more than m elements of P_1. This means that $\underline{<S>}$ consists of the 4-element subset $\underline{<S>} \cap A$, an m-element subset of P_2 and more than m elements lying in P_1. hence $\underline{<S>}$ has more than n elements for $n` = n = 14$, hence $\underline{<S>}$ must be equal to $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$. If $n` = 10$ and $n = 14$, then $\underline{<S>}$ has more than 10 elements, i.e. $|\underline{<S>}| = 14$ or 28. If $|\underline{<S>}| = 14$, then $n` = n = 14$, hence by the preceding argument $\underline{<S>}$ must be equal to $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$.

For the last case $\underline{<S>} \cap A = \{e, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ and for the same reason of the above case the set $\underline{<S>} - A$ contains $\{a_{s_1}, a_{s_2}, \dots, a_{s_m}\} \subseteq P^*_1$ and $\{b_{r_1}, b_{r_2}, \dots, b_{r_m}\} \subseteq P_2$. Let $\underline{<S>}^*$ be the $\mathbf{STS}(s - 1)$ associated with $\underline{<S>}$. Since $\{a_{s_1}, a_{s_2}, \dots, a_{s_m}\} \cap \{a_i, a_j, a_k\} = \varnothing$, it follows that the subset $\{e, a_{s_1}, a_{s_2}, \dots, a_{s_m}\}$ forms a subsloop of $\underline{<S>}$ and also a subsloop of $\boldsymbol{L}_1$. Since $\boldsymbol{L}_1$ is planar, m must be equal 3. Then the $\mathbf{STS}(s - 1) = \underline{<S>}^*$ associated with $\underline{<S>}$ is an STS(9) containing the blocks $\{b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ and $\{a_{s_1}, a_{s_2}, a_{s_3}\}$. Which means that the triple $\{b_{r_1}, b_{r_2}, b_{r_3}\} \subseteq P_2$ must also be a block of $\underline{<S>}^*$, which contradicts the fact that the construction $\underline{\mathbf{2}} \otimes_\alpha \underline{\boldsymbol{L}_1}$ contains exactly one block, $\{b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ lying completely in P_2. Consequently, we can say that the case $\underline{<S>} \cap A = \{e, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$ is also ruled out. According to the above discussion of all possible cases of $\underline{<S>} \cap A$, we may deduce that the subsloop $\underline{<S>}$ is equal to a sub-**SL**(8) or the whole sloop $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$. This means that for $n = 14$ the sloop $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$ has no sub-**SL**(n`)s for all $n$` satisfying $8 < n\text{`} \leq n$.

Secondly, we have to prove that $\underline{\mathbf{2}} \otimes_\alpha \underline{\boldsymbol{L}_1}$ has no proper congruence. Assume that $\underline{\mathbf{2}} \otimes_\alpha \underline{\boldsymbol{L}_1}$ has a congruence θ, if $|[e]\theta| = 8$ or 4, then $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$ has a sub-**SL**(16), which contradicts the proof of the first part, hence we may say that $[e]\theta = \{e, x\}$. If $[e]\theta \cap A = \{e\}$, then $[A]\theta$ is a sub-**SL**(16), which is impossible. Hence $[e]\theta \cap A = [e]\theta$. Say $[e]\theta = \{e, a_i\} \subseteq A$ and suppose that $\{a_j, a_r, a_s\}$ is a block such that $\{a_j, a_r, a_s\} \cap \{a_i, a_j, a_k\} = \{a_j\}$ for $i \neq j$, so we have $[e]\theta \cup [a_j]\theta \cup [a_r]\theta \cup [a_s]\theta = \{e, a_i, a_j, a_i \underline{\cdot} a_j, a_r, a_i \underline{\cdot} a_r, a_s, a_i \underline{\cdot} a_s\} = \{e, a_i, a_j, b_{\alpha(k)}, a_r, a_1, a_s, a_h\}$, where $a_1 = a_i \underline{\cdot} a_r$ and $a_h = a_i \underline{\cdot} a_s$. But $b_{\alpha(k)} \underline{\cdot} a_r = b_v \neq b_{\alpha(k)}$ contradicts the fact that $[e]\theta \cup [a_j]\theta \cup [a_r]\theta \cup [a_s]\theta$ is an 8-element subsloop.

Now suppose that $[e]\theta = \{e, b_{\alpha(i)}\} \subseteq A$ and assume that $\{a_j, b_r, b_s\}$ is a block such that $\{a_j, b_r, b_s\} \cap A = \{a_j\}$ for $i \neq j$. So we have $[e]\theta \cup [a_j]\theta \cup [b_r]\theta \cup [b_s]\theta = \{e, b_{\alpha(i)}, a_j, b_{\alpha(i)} \underline{\cdot} a_j, b_r, b_{\alpha(i)} \underline{\cdot} b_r, b_s, b_{\alpha(i)} \underline{\cdot} b_s\} = \{e, b_{\alpha(i)}, a_j, a_k, b_r, a_1, b_s, a_h\}$, where $b_{\alpha(i)} = a_j \underline{\cdot} a_k$, $a_1 = b_{\alpha(i)} \underline{\cdot} b_r$ and $a_h = b_{\alpha(i)} \underline{\cdot} b_s$. If $a_j \cdot a_l = a_h$, then $a_k \cdot a_l \notin [e]\theta \cup [a_i]\theta \cup [b_r]\theta \cup [b_s]\theta$ contradicts the fact that $[e]\theta \cup [a_i]\theta \cup [b_r]\theta \cup [b_s]\theta$ must be a sub-**SL**(8).

Now, assume that $[e]\theta = \{e, b\} \subseteq A$ and suppose that $\{l, m, n\}$ is a line in X such that $\{\alpha(l), \alpha(m), \alpha(n)\}$ is not a line in X, then $[e]\theta \cup [a_l]\theta \cup [a_m]\theta \cup [a_n]\theta = \{e, b, a_l, b_{\alpha(l)}, a_m, b_{\alpha(m)}, a_n, b_{\alpha(n)}\}$. But according to Lemma 3, the set $\{e, b, a_l, b_{\alpha(l)}, a_m, b_{\alpha(m)}, a_n, b_{\alpha(n)}\}$ does not form an **SL**(8). This means that $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$ has no congruence θ with $[e]\theta = \{e, x\}$, which implies that the constructed $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$ is a semi-planar **SL**($2n$) for $n = 14$ and for all possible r satisfying $0 < r < (n-1)(n-2)/6$. The proof is complete. □

According to Theorem 5 and Theorem 6, we may say that for $n = 14$ and $r = 26$, the constructed sloop **SL**($2n$) = $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$ has $r = (n - 1)(2n - 4)/12$ sub-**SL**(8)s passing through the sub-**SL**(2) = $\{e, b\}$. But $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$ has no sub-**SL**(n)s. So $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$ has exactly one proper congruence θ with $[e]\theta = \{e, b\}$. This means that the constructed sloop **SL**($2n$) = $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$ for $n = 14$ is subdirectly irreducible having only one proper homomorphic image isomorphic to the sub-**SL**(n). Note that the subdirectly irreducible **SL**($2n$) = $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}_1}$ differs from the subdirectly irreducible **SL**($2n$) = $\mathbf{2}\otimes_\alpha\boldsymbol{L}_1$ given in section 3.

5 Examples for each possible class of SL(28)s

We will use the same notations given in section 2 to construct all possible subdirectly irreducible **SL**(28)s, and so also all possible simple **SL**(28)s, as follows:

Let $\boldsymbol{L}_1 = (P_1 = P^*_1 \cup \{e\}; \cdot, e)$ be a planar sloop, in which the 1-factorization $\boldsymbol{F(L_1)}$ have only sub-1-factorizations on the 4-element subsloops. And let $\boldsymbol{T_1} = (P^*_1; B_1)$ be the corresponding **STS**(n - 1), where $P^*_1 = \{a_1, a_2, \dots, a_{n-1}\}$. Consider the set $P_2 = \{b, b_1, \dots, b_{n-1}\}$ such that $P_1 \cap P_2 = \varnothing$.

The constructed **STS**($2n$ - 1) = $\mathbf{2}\otimes_\alpha \boldsymbol{T_1}$ is defined by ($P^* = P^*_1 \cup P_2$; $B = B_1 \cup B_{12}$), where $B_{12} = \{\{a_i, b_j, b_k\} : b_j b_k \in G_{\alpha(i)}\}$ [11]. The associated **SL**($2n$) $\mathbf{2} \otimes_\alpha \boldsymbol{L_1} = (P = P^* \cup \{e\}; \cdot, e)$ has always the sub-**SL**(n) = $\boldsymbol{L_1}$ for each permutation α. So $\boldsymbol{L_1}$ is always normal of $\mathbf{2}\otimes_\alpha\boldsymbol{L_1}$.

By choosing a block $\{a_i, a_j, a_k\} \in B_1$ and interchanging a sub-**STS**(7) with an isomorphic copy on the set $A^* = \{a_i, a_j, a_k, b, b_i, b_j, b_k\}$ in the **STS**($2n$ - 1) = $\mathbf{2}\otimes_\alpha\boldsymbol{T_1}$, we get again a triple system $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{T_1}} = (P^*; B - H \cup R)$ in which the associated **SL**($2n$) = $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L_1}}$ is a simple sloop.

In [4], we have verified all classes of nonsimple **SL**(20)s, six classes of **SL**(20)s (**STS**(19)s) having one sub-**SL**(10) (sub-**STS**(9)) and r sub-**SL**(8)s (sub-**STS**(7)s) for r = 0, 1, 2, 3, 4 or 6. Also, we have shown in [4] that there are 5 classes of simple **SL**(20)s (**STS**(19)s) having r sub-**SL**(8)s (sub-**STS**(7)s) for r = 1, 2, 3, 4, 6, but no sub-**SL**(10) (sub-**STS**(9)).

In section 3, we have verified all classes of nonsimple **SL**(28)s, nine classes of **SL**(28)s (**STS**(27)s) having one sub-**SL**(14) (sub-**STS**(13)) and r sub-**SL**(8)s (sub-**STS**(7)s) for r = 0, 1, 2, 3, 4, 5, 8, 11, 16. And in section 4, we have shown that there are 8 classes of simple **SL**(28)s (**STS**(27)s) having r sub-**SL**(8)s (sub-**STS**(7)s) for r = 1, 2, 3, 4, 5, 8, 11, 16 but no sub-**SL**(14) (sub-**STS**(13)). For r = 26, there are two classes of subdirectly irreducible **SL**(28)s, one of them is the direct product **SL**(14) × **SL**(2) and the others are **SL**(28)s having only one normal sub-**SL**(2), but no sub-**SL**(14).

It is well-known that there are exactly two non isomorphic **STS**(13)s [11]. Let $(P^*_1; B_1)$ be an **STS**(13). The two **STS**(13)s $(P^*_1; B_1)$ and $(P^*_1; B`_1)$ can be described as in the following:

The set of points $P^*_1 = \{a_0, a_1, \dots, a_{12}\}$, where the numbers 0, 1, … 12 are elements of ♦$_{13}$ and the set of blocks B_1 are the subsets $\{1 + i, 2 + i, 5 + i\}$ and $\{1 + i, 6 + i, 8 + i\}$ for $i \in$ ♦$_{13}$ [11]. The other system **STS**(13) = $(P^*_1; B`_1)$ has the same set of points $P^*_1$. The set of blocks $B`_1$ is the same set of blocks B_1 except the four blocks $\{a_1, a_2, a_5\}$, $\{a_5, a_6, a_9\}$, $\{a_1, a_3, a_9\}$ and $\{a_2, a_3, a_6\}$ which replaced by $\{a_1, a_2, a_3\}$, $\{a_1, a_5, a_9\}$, $\{a_3, a_6, a_9\}$ and $\{a_2, a_5, a_6\}$ [11]. In the following we consider the **STS**(13) = (P^*_1; B_1), in fact the same structures can be applied on the other **STS**(13) = (P^*_1; $B`_1$).

Let $\boldsymbol{L}_1$ = ($P_1 = P^*_1 \cup \{e\}$; · , e) be the sloop associated with (P^*_1; B_1). Also, let $P_2 = \{b, b_0, b_1, \dots, b_{12}\}$. The 1-factorizations $\boldsymbol{F}(\boldsymbol{L_1})$ and $\boldsymbol{G}(P_2)$ are defined as in section 2 for n = 14. The **SL**(28) $\mathbf{2}\otimes_\alpha\boldsymbol{L_1}$ = ($P = P^* \cup \{e\}$; · , e) associated with the **STS**(27) $\mathbf{2}\otimes_\alpha\boldsymbol{T_1}$ has the sub-**SL**(14) = $\boldsymbol{L_1}$ for each permutation α. So $\boldsymbol{L_1}$ is always normal in $\mathbf{2}\otimes_\alpha\boldsymbol{L_1}$.

We note that the **STS** = (N; X) ≅ (P^*_1; B_1) for N = {0, 1, … , 12}, so we get the set of lines:

X = {1 2 5, 2 3 6, 3 4 7, 4 5 8, 5 6 9, 6 7 10, 7 8 11, 8 9 12, 9 10 0, 10 11 1, 11 12 2, 12 0 3, 0 1 4, 1 6 8, 2 7 9, 3 8 10, 4 9 11, 5 10 12, 6 11 0, 7 12 1, 8 0 2, 9 1 3, 10 2 4, 11 3 5, 12 4 6, 0 5 7}.

By interchanging the set of blocks:

$$H = \{\{a_1, a_2, a_5\}, \{a_1, b_2, b_5\}, \{a_2, b_1, b_5\}, \{a_5, b_1, b_2\}\}$$

with the set of triples

$$R = \{\{b_1, b_2, b_5\}, \{b_1, a_2, a_5\}, \{b_2, a_1, a_5\}, \{b_5, a_1, a_2\}\}$$

on the set $A^* = \{a_1, a_2, a_5, b, b_1, b_2, b_5\}$, we get the sloop **SL**(28) = $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{L}}_\mathbf{1}$ associated with the constructed triple system $\underline{\mathbf{2}}\otimes_\alpha\underline{\boldsymbol{T}}_\mathbf{1}$ = (P^*; B - $H \cup R$).

The following constructions supplies us with an example for each class of **SL**(28).
Notice that 1 2 5 is a line in X. By choosing the permutation α satisfies that α(1 2 5) = 1 2 5, we get the following classes:

(1) $\alpha_1 = \mathrm{id}_N$; i.e., α_1 transfers each line into the same line in X. The constructed **SL**(28) = $\mathbf{2}\otimes_{\alpha_1}\boldsymbol{L_1}$ is nonsimple and has 26 sub-**SL**(8)s and one sub-**SL**(14); i.e. $\mathbf{2}\otimes_{\alpha_1}\boldsymbol{L_1}$ is isomorphic to **SL**(14) × **SL**(2). Also, the constructed **SL**(28) = $\underline{\mathbf{2}}\otimes_{\alpha_1}\underline{\boldsymbol{L}}_\mathbf{1}$ is nonsimple and has 26 sub-**SL**(8)s but no sub-**SL**(14). This sloop $\underline{\mathbf{2}}\otimes_{\alpha_1}\underline{\boldsymbol{L}}_\mathbf{1}$ is subdirectly irreducible having exactly one proper homomorphic image ≅ **SL**(14).

(2) α_2 = (39) transfers only the 16 lines {1 2 5, 4 5 8, 6 7 10, 7 8 11, 10 11 1, 11 12 2, 0 1 4, 1 6 8, 5 10 12, 6 11 0, 7 12 1, 8 0 2, 9 1 3, 10 2 4, 12 4 6, 0 5 7} into lines. The constructed **SL**(28) = $\mathbf{2}\otimes_{\alpha_2}\boldsymbol{L_1}$ is nonsimple and has 16 sub-**SL**(8)s and one sub-**SL**(14) and the constructed **SL**(28) = $\underline{\mathbf{2}}\otimes_{\alpha_2}\underline{\boldsymbol{L}}_\mathbf{1}$ is simple and has 16 sub-**SL**(8)s but no sub-**SL**(14).

(3) α_3 = (39) (04) transfers 11 lines into lines; the set of the 11 lines is {1 2 5, 6 7 10, 7 8 11, 10 11 1, 11 12 2, 0 1 4, 1 6 8, 5 10 12, 7 12 1, 9 1 3}.

(4) α_4 = (347)(89) transfers only 8 lines into lines, it preserves the 7 lines 1 2 5, 3 4 7, 8 9 12, 10 11 1, 11 12 2, 6 11 0, 5 10 12 and transfers the line 4 9 11 into the line 7 8 11.

(5) α_5 = (347) (89) (10 11) transfers 5 lines. α_5 preserves the set of lines {1 2 5, 3 4 7, 8 9 12, 10 11 1} and transfers the line 3 8 10 into the line 4 9 11.

(6) α_6 = (1 2) (3 4 6 8 11) transfers only the 4 lines {1 2 5, 9 10 0, 5 10 12, 0 5 7} into lines.

(7) α_7 = (1 2) (3 4 6 8 11) (0 7) transfers only the 3 lines {1 2 5, 5 10 12, 0 5 7} into lines.

(8) α_8 = (1 2) (4 6 8 11) (0 9 10) transfers only the 2 lines {1 2 5, 9 10 0} into lines.

(9) α_9 = (1 2) (3 4 6 8 11) (0 12) transfers only the line 1 2 5 into a line (into itself).

The constructed **SL**(28) = $\mathbf{2}\otimes_\alpha \boldsymbol{L}_1$ is non-simple and has one sub-**SL**(14) and r sub-**SL**(8)s. The constructed **SL**(28) = $\underline{\mathbf{2}}\otimes_\alpha \underline{\boldsymbol{L}_1}$ is simple and has r sub-**SL**(8)s but no sub-**SL**(14), where

r = 16, 11, 8, 5, 4, 3, 2, 1 for $\alpha = \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9$, respectively.

(10) α_{10} = (1 2) (3 5) (6 8 9 11) does not transfer any line into a line. The constructed **SL**(28) = $\mathbf{2}\otimes_\alpha \boldsymbol{L}_1$ is nonsimple and has one sub-**SL**(14), but no sub-**SL**(8)s.

The subsloops of the above examples mentioned in both items for n = 14 are the normal **SL**(n) = $\boldsymbol{L}_1$ and the sub-**SL**(8)s determined by the set $\{e, a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$, where $\{i, j, k\}$ and $\{\alpha(i), \alpha(j), \alpha(k)\}$ are lines belonging to the set X.

Similarly, we can also construct all possible classes of subdirectly irreducible **SL**(2n)s = $\mathbf{2}\otimes_\alpha \boldsymbol{L}_1$ and simple **SL**(2n)s =$\underline{\mathbf{2}}\otimes_\alpha \underline{\boldsymbol{L}_1}$ for each possible $n > 14$. If we begin with a planar **SL**(n) = $\boldsymbol{L}_1$ satisfying that all proper sub-1-factorizations of the 1-factorization $\boldsymbol{F}(\boldsymbol{L}_1)$ are defined only on the sub-**SL**(4)s of $\boldsymbol{L}_1$, then one can apply Theorems 3, 4, 5, 6 to determine all possible classes of **SL**(2n)s for each $n > 14$. One need only determine the values of the number r of the sub-**SL**(8)s of the constructed **SL**(2n) for each cardinality n.

6 Classification of SL(2*n*)s for small value of 2*n* (2*n* = 16, 20, 28)

In this section we summarize the classes of **SL**(n)s for each possible $n < 32$. All classes of **SL**(32)s are determined in [3].

It is well-known that there are only Boolean **SL**(n) for n =2, 4 and 8. Also, there are only simple planar **SL**(n)s for n = 10 and 14 and simple **SL**(n)s for n = 22 and 26. For the cardinalities 16, 20, and 28, we have the following classes:

For 2n =16 [2, 5, 11], it is well-known that the possible nontrivial subsloops are only the sub-**SL**(8)s and the classes of **SL**(16)s are:

Type of **SL**(16)s	Simple **SL**(16)s	Nonsimple **SL**(16)s			
Number of sub-**SL**(8)s	0	1	3	7	15
Interested property	Planar	Subdirectly irreducible			Boolean **SL**$(2)^4$

For $2n$ =20 [4], the possible nontrivial subsloops are sub-**SL**(8)s and sub-**SL**(10). We note that there is only one **SL**(10). According to the results given in [4], all classes of **SL**(20)s are:

Type of **SL**(20)s	Simple **SL**(20)s						Nonsimple **SL**(20)s							
Number of sub-**SL**(10)s	0						1	1	1	1	1	1	0	1
Number of sub-**SL**(8)s	0	1	2	3	4	6	0	1	2	3	4	6	12	12
Interested property	Planar	Semi-planar					Subdirectly irreducible having only one proper congruence.							**SL**(10) × **SL**(2)

These classes are in complete agreement with the results determined by computer programs for **STS**(19)s given in [10].

For $2n$ =28, the possible nontrivial subsloops are sub-**SL**(8)s, sub-**SL**(10)s and sub-**SL**(14)s. According to Lemma 1, an **SL**(28) with sub-**SL**(14) has no sub-**SL**(10)s. It is well-known that there are two distinct **SL**(14)s [11]. For each **SL**(14), the classes of **SL**(28)s are:

Type of **SL**(28)s	Simple **SL**(28)s									Nonsimple **SL**(28)s										
Number of sub-**SL**(14)s	0									1	1	1	1	1	1	1	1	1	0	1
Number of sub-**SL**(8)s	0	1	2	3	4	5	8	11	16	0	1	2	3	4	5	8	11	16	26	26
Interested property	Planar	Semi-planar								Subdirectly irreducible having only one proper congruence.										**SL**(14) × **SL**(2)

The above classes of **SL**(28)s are all possible classes having no sub-**SL**(10)s. Of course there are other classes of **SL**(28)s (**STS**(27)s) having sub-**SL**(10)s (sub-**STS**(9)s), for example the direct product **STS**(9) × **STS**(3). So, we are faced with the question about the classes of **SL**(28)s (**STS**(27)s) having sub-**SL**(10)s (sub-**STS**(9)s).

In general for $n \geq 16$, if we begin with a planar **SL**(n) = $\boldsymbol{L_1}$ having a 1-factorization $\boldsymbol{F}(\boldsymbol{L_1})$ on $\boldsymbol{L_1}$ satisfying that proper sub-1-factorizations of $\boldsymbol{F}(\boldsymbol{L_1})$ exist only on the sub-**SL**(4)s of $\boldsymbol{L_1}$**,** we may construct all possible classes that can be constructed by the doubling construction of triple systems; namely, the constructions **SL**(2n) = **2** $\otimes_\alpha$ $\boldsymbol{L_1}$ and **SL**(2n) = $\underline{\mathbf{2}}$ $\otimes_\alpha$ $\underline{\boldsymbol{L_1}}$; i.e., we can construct all possible classes of nonsimple **SL**(2n)s having a planar sub-**SL**(n) or a normal sub-**SL**(2), and all possible classes of simple **SL**(2n)s having only sub-**SL**(8)s.

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