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Structure analyses of the perimeters of primitive Pythagorean triples

J.V. Leyendekkers[1] and A.G. Shannon[2]

[1] The University of Sydney, 2006 Australia

[2] Faculty of Engineering & IT, University of Technology Sydney, 2007 Australia

Abstract. Structural analysis (via the modular rings Z_4, Z_6) shows that the Perimeters, Pr, of primitive Pythagorean Triples (pPts) do not belong to simple functions. However, the factors *x*, (*x*+*y*) of the perimeter do, and the number of pPts in a given interval can be estimated from this. When *x* is prime, the series for (*x*+*y*) is complete and the associated pPts are one third of the total. When *x* is composite, members of the series for (*x*+*y*) are invalid when common factors with *x* occur. These members are not associated with pPts. When $3|(x+y)$, $\text{Pr} \in \bar{3}_6$, while if $3 \nmid (x+y), \text{Pr} \in \{\bar{1}_6, \bar{3}_6\}$. Class $\bar{3}_6$ dominates in the distribution.

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1 Introduction

We have recently analysed the structural characteristics of the major components of primitive Pythagorean triples (pPts), using the modular rings Z_4, Z_6 (Tables 1 and 2). This permitted an estimate of the number of pPts in a given range and the constraints which prevent the formation of pPts [4].

Row	$f(r)$	$4r_0$	$4r_1+1$	$4r_2+2$	$4r_3+3$
	Class	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$
0		0	1	2	3
1		4	5	6	7
2		8	9	10	11
3		12	13	14	15
4		16	17	18	19
5		20	21	22	23
6		24	25	26	27
7		28	29	30	31

Table 1: Rows of Z_4

This study also showed the link between the number of primes in $\bar{1}_4$ and the number of pPts in the range. In this paper we consider the perimeter, Pr, of pPts; that is, for

$$c^2 = a^2 + b^2$$
$$\text{Pr} = a + b + c.$$

This quantity has been used to develop a theoretical estimate of the number of pPts in a given range [1]. In addition, other new characteristics of the major component and (*x*,*y*) couples will be illustrated, where :

$$\text{Pr} = 2x(x + y).$$

Row	$f(r)$	$6r_1 - 2$	$6r_2 - 1$	$6r_3$	$6r_4 + 1$	$6r_5 + 2$	$6r_6 + 3$
	Class	$\bar{1}_6$	$\bar{2}_6$	$\bar{3}_6$	$\bar{4}_6$	$\bar{5}_6$	$\bar{6}_6$
0		-2	-1	0	1	2	3
1		4	5	6	7	8	9
2		10	11	12	13	14	15
3		16	17	18	19	20	21
4		22	23	24	25	26	27
5		28	29	30	31	32	33
6		34	35	36	37	38	39
7		40	41	42	43	44	45

Table 2: Rows of Z_6

2. Factors of the Perimeter of a pPt

The major component $c \in \bar{1}_4$, as the integers in this Class in Z_4 may form a sum of squares, whereas integers in $\bar{3}_4$ cannot [2,4]. We have used Z_4 to illustrate that $c \in \bar{1}_4$, and this follows simply since $\bar{3}_4\,\bar{2}_4$ have no squares (Table 1) so that $\bar{1}_4 + \bar{0}_4 = \bar{1}_4$ or $x^2 + y^2 = c$.
However, analysis of pPts is best done using Z_6 (Table 2). This ring allows odd integers, *N*, with 3|*N* to be isolated in Class $\bar{6}_6$ (Table 2). This is useful as *c* cannot have a factor of 3. Thus we need only consider the two Classes $\bar{2}_6, \bar{4}_6$ (Tables 1-3).

Class	**Row in Z_6**	**Row in Z_4**
$\bar{2}_6$	R_2 odd	r_1 odd or even
$\bar{4}_6$	R_4 even	$3 \mid r_1$

Table 3: Integers from Class $\bar{1}_4 \subset Z_4$ transferred to Z_6

The factor, *x*, of Pr is even or odd, but (*x* + *y*) is odd. When *x* is constant (*x* + *y*) follows a regular sequence (Table 4,5) with

$$x+y_0=\begin{cases}x+2, & x\,odd,\\ x+1, & x\,even.\end{cases}$$

The largest y is $(x-1)$, since $x>y$. Only when x is a prime or equal to 2^n is the sequence complete (Tables 4,5).

x	**Range of (x+y_i), i:0-n**	**Number of elements of sequence**	**Range of *c***
3	5	1	13
5	7-9	2	29-41
7	9-13	3	53-85
11	13-21	5	125-221
13	15-25	6	173-313
17	19-33	8	293-685
19	21-37	9	365-685
23	25-45	11	533-1013
29	31-57	14	845,857,877.905.941
		6 (to 1013)	985-1625
31	33-61	15	965,977,997 to
		3 (to (1013)	1861
37	39-73	18	1373-2665
		0 (to 1013)	
	TOTAL:	54 (to 1013)	

Table 4: Characteristics of factors of pPt perimeter, Pr =2x(x+y), y_0=2, y_n=x-1

x	**Range of (x+y_i), i:0-n**	**Missing elements of sequence (x+y_i)**	**Number of elements of sequence**	**Range of *c***
2	3	-	1	5
4	5-7	-	2	17-25
6	7-11	9	2	37-61
8	9-15	-	4	65-113
9	11-17	15	3	85-145
10	11-19	15	4	101-181
12	13-23	15,21	4	145-265
14	15-27	21	6	197-365
15	17-29	21, 25, 27	4	229-421
16	17-31	-	8	257-481
18	19-35	21, 27, 33	6	325-613
20	21-39	25, 35	8	401-761
21	23-41	27, 33, 35, 39	6	445-841
22	23-43	33	10	485-925
24	25-47	27, 33, 39, 45	8	577, 601, 625, 697, 745, 865,
			7 (to 1013)	937 to 1105

25	27-49	35,45	10	629, 641, 661, 689, 769, 821, 881, 949 to 1201
			8 (to 1013)	
26	27-51	39	12	677, 685, 701, 725, 757, 797, 901, 965, 1037 to 1301
			8 (to 1013)	
27	29-53	33, 39, 45, 51	8	733, 745, 793, 829, 925, 985, 1053 to 1405
			6 (to 1013)	
28	29-55	35, 49	11	785, 793, 809, 865, 905, 953, 1009 to 1513
			7 (to 1013)	
30	31-59	33, 35, 39, 45, 51, 55, 57	7	901, 949, 1021 to 1741
			2 (to 1013)	
32	33-63	-	16	1024-1985
			0 (to 1013)	
	TOTAL:		106 (to 1013)	

Table 5: Characteristics of factors of pPt perimeter, Pr =2*x*(*x*+*y*),
y_0=1,(*x* even), y_0=2 (*x* odd), y_n=x-1

Otherwise, when *x* and (*x*+*y*) have a common factor the (x,*y*) couple cannot form a pPt. Obviously, when $x = 2^n$ or a prime there cannot be a common factor with (*x*+*y*). This means that primes can be distinguished from the odd composites (Tables 4 and 5). The Class of Pr will depend on the classes of *x* and *y* (Table 6).

x	***y***	**Pr=2*x*(*x*+*y*)**	**Examples**
$\bar{6}_6$ ($6r_6$+3)	$\bar{1}_6$ ($6r_1$-2)	$\bar{3}_6$ $6r_3$(r_3 odd)	*x*=9, *y*=4; Pr =234=6×39
$\bar{2}_6$ ($6r_2$-1)	$\bar{1}_6$ ($6r_1$-2)	$\bar{3}_6$ $6r_3$ (r_3 odd)	*x*=11, *y*=4; Pr=330=6×55
$\bar{4}_6$ ($6r_4$+1)	$\bar{1}_6$ ($6r_1$-2)	$\bar{1}_6$ r_1 even	*x*=7, *y*=4; Pr=154=6×26-2
$\bar{6}_6$	$\bar{3}_6$	invalid	since 3|*x*, 3|*y*
$\bar{2}_6$	$\bar{3}_6$	$\bar{5}_6$ ($6r_5$+2) (r_5 even)	*x*=11, *y*=6; Pr=374=6×62+2
$\bar{4}_6$	$\bar{3}_6$	$\bar{5}_6$ (r_5 even)	*x*=7, *y*=6; Pr=182=6×30+2
$\bar{6}_6$	$\bar{5}_6$	$\bar{3}_6$ (r_3 odd)	*x*=3, *y*=2; Pr=30=6×5
$\bar{2}_6$	$\bar{5}_6$	$\bar{1}_6$ (r_1 even)	*x*=17, *y*=14; Pr=1054=6×176-2
$\bar{4}_6$	$\bar{5}_6$	$\bar{3}_6$ (r_3 odd)	*x*=19, *y*=8; Pr=1026=6×171

Table 6: Class distribution. (When Pr$\in\bar{3}_6$, row always odd; Pr$\in\bar{1}_6,\bar{5}_6$ row always even)

Note that when $Pr \in \bar{3}_6$, then 3|Pr, and the row is always odd. When $Pr \{\bar{1}_6, \bar{5}_6\}$, the row is always even. This gives a check on Pr. The reader might like to investigate why integers in these Classes in alternate rows (that is, even rows in $\bar{3}_6$, etc) cannot be Pr values. Use Class function and *a,b,c* as f(*x,y*).

Class $\bar{3}_6$ has twice as many Pr values as either $\bar{1}_6$ or $\bar{5}_6$. Thus the Pr commonly has a factor of 3 (Table 7).

The distribution among the three Classes differs for primes and odd composites (Table 8), although Class $\bar{3}_6$ dominates for both, as predicted (Table 6).

c	509	521	541	557	569	577	593	601	613	617	641	653
Pr	1188	1240	1302	1254	1320	1200	1426	1392	1260	1330	1450	1540
Class	$\bar{3}_6$	$\bar{1}_6$	$\bar{3}_6$	$\bar{3}_6$	$\bar{3}_6$	$\bar{3}_6$	$\bar{1}_6$	$\bar{3}_6$	$\bar{3}_6$	$\bar{1}_6$	$\bar{1}_6$	$\bar{1}_6$

c	661	673	677	701	709	733	757	761	769	773	797	809
Pr	1550	1610	1404	1612	1628	1566	1820	1560	1850	1716	1924	1848
Class	$\bar{5}_6$	$\bar{5}_6$	$\bar{3}_6$	$\bar{1}_6$	$\bar{5}_6$	$\bar{3}_6$	$\bar{5}_6$	$\bar{3}_6$	$\bar{5}_6$	$\bar{3}_6$	$\bar{1}_6$	$\bar{3}_6$

c	821	829	853	857	877	881	929	937	941	953	977	997
Pr	1950	1998	1886	1914	2030	2050	1978	2064	2262	2296	2170	2294
Class	$\bar{3}_6$	$\bar{3}_6$	$\bar{5}_6$	$\bar{3}_6$	$\bar{5}_6$	$\bar{1}_6$	$\bar{1}_6$	$\bar{3}_6$	$\bar{3}_6$	$\bar{1}_6$	$\bar{1}_6$	$\bar{5}_6$

Table 7: Perimeter values and Classes

Distribution of Pr among the three Classes for even integers in Z_6			
Classes	$\bar{1}_6$	$\bar{3}_6$	$\bar{5}_6$
Primes (*c*)	31%	47%	22%
Composites (*c*)	16%	56%	28%

Table 8: Range of *c*, 500-1000

When the factor *x* of Pr has 3|*x*, then *c* never falls in Class $\bar{2}_6$. If $3 \nmid y$, then $c \in \bar{4}_6$. These results follow from the structure of Z_6 (Table 2); Classes $\bar{5}_6$ and $\bar{2}_6$ have no even powers.

3 Maximum values of *c* in (*x*+*y*) sequences

It is of interest to note that the *c* maximum has a right-end-digit (RED) of 1,3,5. Why is c_{max}* not equal to 7 or 9? The asterisk indicates RED. Since

$$(x^2)^* = 1, 5 \text{ or } 9 \ (x \text{ odd})$$

and

$$(y^2)^* = 0,4 \text{ or } 6 \ (y \text{ even}),$$

then normally 9, 7 would apply (Table 9). However, *y* is restricted to (*x*-1), so that

$$x^2 + y^2 = 2x^2 - 2x + 1.$$

$(y^2)^*\downarrow$ $(x^2)^*\rightarrow$	1	5	9
0	1	5	9
4	5	9	3
6	7	1	5

Table 9: REDs for $(x^2)^*$ and $(y^2)^*$

$(2x^2)^*$	2	8	0	8	2
$-(2x^2)^*$	-2	-6	0	-4	-8
1*	1	1	1	1	1
f(x)	1	3	1	5	5

Table 10: 1,3,5 constraint

4 Number of pPts in a given range

We have recently discussed various ways of predicting the number of pPts in a given range [4]. In the range to 1013 there are 160 pPts (Tables 4,5). When x is a prime the contribution is 54 (Table 4) or one third of the total.

The perimeter does not increase with c in a simple manner (Table 7). If $c_2 > c_1$, then $\mathrm{Pr}_2 \ngtr \mathrm{Pr}_1$ necessarily. In fact, $\mathrm{Pr}_1 > \mathrm{Pr}_2$ regularly (Table 7). Hence, the prediction of the number of pPts in a given interval, M, using $M \propto \mathrm{Pr}$ [1] cannot, in general, be very accurate. For 0-1000, if 2294 is taken, the result is 160, but the same value would be found for 0-950! When the class and row parity of Pr are considered, simpler functions are possible.

5 Final comments

Odd integers that equal a sum of squares have been known since the time of Fermat to equal $(4r_1+1)$ always; that is, in terms of the modular ring Z_4 these integers always fall in Class $\bar{1}_4$. We have investigated various consequences of this fact, and used modular rings to analyse the underlying structures which permit easier predictions of x,y [2,3,4].

With x constant, $(x+y)$ conforms to a regular sequence with the constraint that members which have common factors for x are omitted. This allows x,y values to be predicted readily for a given range. Prediction of the number of pPts in a given range follows from the x, $(x+y)$ characteristics, but the perimeter does not seem to be a suitable quantity for predicting the number of pPts.

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