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New results on some multiplicative functions

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Abstract

The paper is a continuation of [1]. The considerations are over the class of multiplicative functions with strictly positive values and more precisely, over the pairs (f, g) of such functions, which have a special property, called in the paper property $\mathbb{S}$. For every such pair (f, g) and for every composite number $n > 1$, the problem of finding the maximum and minimum of the numbers $f(d)g\left(\frac{n}{d}\right)$, when d runs over all proper divisors of n, is completely solved. Since some classical multiplicative functions like Euler's totient function φ, Dedekind's function ψ, the sum of all divisors of m, i.e. $\sigma(m)$, the number of all divisors of m, i.e. $\tau(m)$, and $2^{\omega(m)}$ (where $\omega(m)$ is the number of all prime divisors of m) form pairs having property $\mathbb{S}$, we apply our results to these functions and also resolve the questions of finding the maximum and minimum of the numbers $\varphi(d)\sigma\left(\frac{n}{d}\right)$, $\varphi(d)\psi\left(\frac{n}{d}\right)$, $\tau(d)\sigma\left(\frac{n}{d}\right)$, $2^{\omega(d)}\sigma\left(\frac{n}{d}\right)$, where d runs over all proper divisors of n. In addition some corollaries from the obtained results, concerning unitary proper divisors, are made. Since many other pairs of multiplicative functions (except the considered in the paper) have property $\mathbb{S}$, they may be investigated in similar manner in a future research.

Keywords: multiplicative functions, divisors, proper divisors, unitary divisors, proper unitary divisors, prime numbers, composite number

AMS Subject Classification: 11A25

Used Denotations: $\mathbb{Z}^+$ - the set of all non-negative integers; $\mathbb{N}$ - the set of all positive integers; $\mathbb{P}$ - the set of all prime numbers; for a given $n \in \mathbb{N}$ D_n^* denotes the set of all proper divisors of n, i.e. different than 1 and n; D_n^{**}- the set of all proper unitary divisors of n (i.e. different than 1 and n)(see [2]); for $n > 1$ $\omega(n)$ denotes the number of all prime divisors of n ($\omega(1) \stackrel{\text{def}}{=} 0$); for $a, b \in \mathbb{N}$ $\gcd(a, b)$ denotes the greatest common divisor of a and b; for $p \in \mathbb{P}$ $\text{ord}_p n$ denotes the largest exponent k for which p^k is a divisor of n.

1 Introduction

The present paper is a continuation of the research from [1]. We remind that an artihmetic function F is said to be multiplicative if for every $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$ it is fulfilled

$$F(ab) = F(a)F(b)$$

Therefore, $F(1) = 1$ if $F \not\equiv 0$.

Some classical examples of multiplicative functions that have an importnat meaning in Number Theory are Euler's totient function (the function φ), Dedekind's function (the function ψ), sum of all divisors of a positive integer (the function σ) and the number of all divisors of a positive integer (the function τ). When $n > 1$ these functions admit the following multiplicative representations:

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right);$$

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right);$$

$$\sigma(n) = \prod_{p|n} \frac{p^{1+\mathrm{ord}_p n} - 1}{p - 1};$$

$$\tau(n) = \prod_{p|n} (1 + \mathrm{ord}_p n),$$

where p runs over all prime divisors of n.

Below we shall consider only the class $\mathbb{M}$ of all multiplicative functions with strictly positive values. Our investigation is based on some pairs of multiplicative functions from the class $\mathbb{M}$ which have a special property (called in the paper property $\mathbb{S}$). For such pairs we completely decide the question about finding the $\max\limits_{d \in \mathbb{D}_n^*} \left\{f(d)g\left(\frac{n}{d}\right)\right\}$ and $\min\limits_{d \in \mathbb{D}_n^*} \left\{f(d)g\left(\frac{n}{d}\right)\right\}$ when $n > 1$ is a composite number. Since some pairs of classical multiplicative functions (like (φ, σ), (φ, ψ), (τ, σ),$(2^{\omega(m)}, \sigma)$) have property $\mathbb{S}$, we apply our theorems to them to obtain some new results.

2 Preliminary results

Definition. *Let $f, g \in \mathbb{M}$. We say that the ordered pair (f, g) has the property $\mathbb{S}$ when one of the following two cases is fulfilled:*

(i) $\forall p \in \mathbb{P}$ & $\forall m \in \mathbb{Z}^+$

$$H_{p,m}^{f,g}(k) \stackrel{\text{def}}{=} f(p^k)g(p^{m-k}) \tag{1}$$

is an increasing function (not necessarily strictly) with respect to $k \in [0, m] \cap \mathbb{Z}^+$

(ii) $\forall p \in \mathbb{P}$ & $\forall m \in \mathbb{Z}^+$ *the function* $H_{p,m}^{f,g}$ *from* (1) *is a decreasing function (not necessarily strictly) with respect to* $k \in [0, m] \cap \mathbb{Z}^+$

In [1] the following result is established (see [1, Theorem 5]):

Theorem 1. *Let $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (ii)). If $n > 1$ is a composite number, then:*

$$\max_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = \max_p \left\{ f(p) g\left(\frac{n}{p}\right) \right\} \tag{2}$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = \min_p \left\{ g(p) f\left(\frac{n}{p}\right) \right\} \tag{3}$$

where p runs over all prime divisors of n.

Remark 1. *If $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (i)), then the pair (g, f) has the property $\mathbb{S}$ (satisfying (ii)).*

So we are able to apply Theorem 1 by exchanging the places of f and g. According to Remark 1 the following result is true (see also [1, Theorem 5]):

Theorem 2. *Let $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (i)). If $n > 1$ is a composite number, then:*

$$\max_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = \max_p \left\{ g(p) f\left(\frac{n}{p}\right) \right\} \tag{4}$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = \min_p \left\{ f(p) g\left(\frac{n}{p}\right) \right\} \tag{5}$$

where p runs over all prime divisors of n.

3 Main results

Let $f, g \in \mathbb{M}$ and $n > 1$ be a composite number. Let

$$n = \prod_{q \mid n} q^{\alpha_q}, \text{ where } \alpha_q \stackrel{\text{def}}{=} \text{ord}_q n \tag{6}$$

be the canonical prime factorization of n, i.e. q runs over all prime divisors of n. Then if p is a prime divisor of n we have:

$$f(p) g\left(\frac{n}{p}\right) = f(p) g\left(p^{\alpha_p - 1} \prod_{\substack{q \mid n \\ q \neq p}} q^{\alpha_q} \right) = f(p) g\left(p^{\alpha_p - 1}\right) g\left(\prod_{\substack{q \mid n \\ q \neq p}} q^{\alpha_q} \right) =$$

$$= f(p) \frac{g\left(p^{\alpha_p - 1}\right)}{g\left(p^{\alpha_p}\right)} g\left(p^{\alpha_p}\right) g\left(\prod_{\substack{q \mid n \\ q \neq p}} q^{\alpha_q} \right) = f(p) \frac{g\left(p^{\alpha_p - 1}\right)}{g\left(p^{\alpha_p}\right)} g\left(\prod_{q \mid n} q^{\alpha_q} \right) =$$

$$= f(p) \frac{g\left(p^{\alpha_p - 1}\right)}{g\left(p^{\alpha_p}\right)} g(n)$$

From the above and from(2),(3), (4)and (5) we get the **first two main results** of the paper:

Theorem 3. *Let $n > 1$ be a composite number. If $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (ii)), then*

$$\max_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = g(n) \max_p \left\{ f(p) \frac{g\left(p^{-1+\operatorname{ord}_p n}\right)}{g\left(p^{\operatorname{ord}_p n}\right)} \right\} \tag{7}$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = f(n) \min_p \left\{ g(p) \frac{f\left(p^{-1+\operatorname{ord}_p n}\right)}{f\left(p^{\operatorname{ord}_p n}\right)} \right\} \tag{8}$$

where p runs over all prime divisors of n.

Theorem 4. *Let $n > 1$ be a composite number. If $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (i)), then*

$$\max_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = f(n) \max_p \left\{ g(p) \frac{f\left(p^{-1+\operatorname{ord}_p n}\right)}{f\left(p^{\operatorname{ord}_p n}\right)} \right\} \tag{9}$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = g(n) \min_p \left\{ f(p) \frac{g\left(p^{-1+\operatorname{ord}_p n}\right)}{g\left(p^{\operatorname{ord}_p n}\right)} \right\} \tag{10}$$

where p runs over all prime divisors of n.

Let $n > 1$ be a squarefree composite number. In this case, each proper divisor of n is a unitary proper divisor of n. Therefore, $\mathbb{D}_n^*$ coincides with $\mathbb{D}_n^{**}$. Then as a corollary from Theorem 3 and Theorem 4 we obtain:

Theorem 5. *Let $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (ii)). Then for every composite squarefree number $n > 1$ it is fulfilled:*

$$\max_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = g(n) \max_p \left\{ \frac{f(p)}{g(p)} \right\}$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = f(n) \min_p \left\{ \frac{g(p)}{f(p)} \right\}$$

where p runs over all prime divisors of n.

Theorem 6. *Let $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (i)). Then for every composite squarefree number $n > 1$ it is fulfilled:*

$$\max_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = f(n) \max_p \left\{ \frac{g(p)}{f(p)} \right\}$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ f(d) g\left(\frac{n}{d}\right) \right\} = g(n) \min_p \left\{ \frac{f(p)}{g(p)} \right\}$$

where p runs over all prime divisors of n.

As a corollary from Theorem 5 and Theorem 6 we obtain:

Theorem 7. *Let $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (ii)). Then for every composite squarefree number $n > 1$ it is fulfilled:*

$$\max_{d \in \mathbb{D}_n^*} \left\{ \frac{f(d)}{g(d)} \right\} = \max_p \left\{ \frac{f(p)}{g(p)} \right\}$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ \frac{f(d)}{g(d)} \right\} = \frac{f(n)}{g(n)} \min_p \left\{ \frac{g(p)}{f(p)} \right\}$$

where p runs over all prime divisors of n.

Theorem 8. *Let $f, g \in \mathbb{M}$ and the pair (f, g) has the property $\mathbb{S}$ (satisfying (i)). Then for every composite squarefree number $n > 1$ it is fulfilled:*

$$\max_{d \in \mathbb{D}_n^*} \left\{ \frac{f(d)}{g(d)} \right\} = \frac{f(n)}{g(n)} \max_p \left\{ \frac{g(p)}{f(p)} \right\}$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ \frac{f(d)}{g(d)} \right\} = \min_p \left\{ \frac{f(p)}{g(p)} \right\}$$

where p runs over all prime divisors of n.

Other corollaries from Theorem 3 and Theorem 4, are obtained below for the case when $g(m) = m \ \forall m \in \mathbb{N}$:

Theorem 9. *Let $f \in \mathbb{M}$ and the pair (f, g), with $g(m) = m \ \forall m \in \mathbb{N}$, has the property $\mathbb{S}$ (satisfying (ii)). Then for every composite number $n > 1$ it is fulfilled:*

$$\max_{d \in \mathbb{D}_n^*} \left\{ \frac{f(d)}{d} \right\} = \max_p \left\{ \frac{f(p)}{p} \right\}$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ \frac{f(d)}{d} \right\} = \min_p \left\{ \frac{f\left(\frac{n}{p}\right)}{\frac{n}{p}} \right\}$$

where p runs over all prime divisors of n.

Theorem 10. *Let $f \in \mathbb{M}$ and the pair (f, g), with $g(m) = m \ \forall m \in \mathbb{N}$, has the property $\mathbb{S}$ (satisfying (i)). Then for every composite number $n > 1$ it is fulfilled:*

$$\max_{d \in \mathbb{D}_n^*} \left\{ \frac{f(d)}{d} \right\} = \max_p \left\{ \frac{f\left(\frac{n}{p}\right)}{\frac{n}{p}} \right\}$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ \frac{f(d)}{d} \right\} = \min_p \left\{ \frac{f(p)}{p} \right\}$$

where p runs over all prime divisors of n.

We shall apply (7) in the particular case: $f = \varphi,\ g = \sigma$. In this case one may verify that the pair $(\varphi,\ \sigma)$ has the property $\mathbb{S}$ (satisfying (ii)) and obtain:

$$\max_{d\in\mathbb{D}_n^*}\left\{\varphi(d)\sigma\left(\frac{n}{d}\right)\right\} = \sigma(n)\max_p\left\{\varphi(p)\frac{\sigma\left(p^{-1+ord_p n}\right)}{\sigma\left(p^{ord_p n}\right)}\right\}, \tag{11}$$

where p runs over all prime divisors of n.

Since

$$\varphi(p) = p-1;\ \sigma\left(p^{-1+\alpha_p}\right) = \frac{p^{\alpha_p}-1}{p-1};\ \sigma\left(p^{\alpha_p}\right) = \frac{p^{\alpha_p+1}-1}{p-1},$$

where $\alpha_p \stackrel{\text{def}}{=} \text{ord}_p n$, (11) yields:

$$\max_{d\in\mathbb{D}_n^*}\left\{\varphi(d)\sigma\left(\frac{n}{d}\right)\right\} = \sigma(n)\max_p\left\{(p-1)\frac{p^{\alpha_p}-1}{p^{\alpha_p+1}-1}\right\}, \tag{12}$$

where p runs over all prime divisors of n.

Using the identity

$$(p-1)\frac{p^{\alpha_p}-1}{p^{\alpha_p+1}-1} = 1-\frac{1}{p}\left(1+\frac{(p-1)^2}{p^{\alpha_p+1}-1}\right)$$

we may rewrite (12) in the form:

$$\max_{d\in\mathbb{D}_n^*}\left\{\varphi(d)\sigma\left(\frac{n}{d}\right)\right\} = \sigma(n)\left(1-\min_p\left\{\frac{1}{p}\left(1+\frac{(p-1)^2}{p^{1+\text{ord}_p n}-1}\right)\right\}\right),$$

where p runs over all prime divisors of n.

Hence it is proved that:

$$\max_{d\in\mathbb{D}_n^*}\left\{\varphi(d)\sigma\left(\frac{n}{d}\right)\right\} = \sigma(n)\left(1-\min_p\left\{\frac{1}{p}\left(1+\frac{\varphi(p)}{\sigma(p^{\text{ord}_p n})}\right)\right\}\right), \tag{13}$$

where p runs over all prime divisors of n.

Also it is easy to see that the identity:

$$(p-1)\frac{p^{\alpha_p}-1}{p^{\alpha_p+1}-1} = \left(1-\frac{1}{p}\right)\left(1-\frac{1}{\sigma(p^{\text{ord}_p n})}\right)$$

holds. From the above identity and (12) it follows that:

$$\max_{d\in\mathbb{D}_n^*}\left\{\varphi(d)\sigma\left(\frac{n}{d}\right)\right\} = \sigma(n)\max_p\left\{\left(1-\frac{1}{p}\right)\left(1-\frac{1}{\sigma(p^{\text{ord}_p n})}\right)\right\} \tag{14}$$

where p runs over all prime divisors of n.

Since (13) and (14) are true, we may consider as proved the following **third main result** of the present paper:

Theorem 11. *Let $n > 1$ be a composite number. Then the following two representations are fulfilled:*

$$\max_{d\in\mathbb{D}_n^*}\left\{\varphi(d)\sigma\left(\frac{n}{d}\right)\right\} = \sigma(n)\left(1-\min_p\left\{\frac{1}{p}\left(1+\frac{\varphi(p)}{\sigma(p^{\mathrm{ord}_p n})}\right)\right\}\right) \tag{*}$$

$$\max_{d\in\mathbb{D}_n^*}\left\{\varphi(d)\sigma\left(\frac{n}{d}\right)\right\} = \sigma(n)\max_p\left\{\left(1-\frac{1}{p}\right)\left(1-\frac{1}{\sigma(p^{\mathrm{ord}_p n})}\right)\right\}, \tag{**}$$

where p runs over all prime divisors of n.

Corollary 1. *Let $n > 1$ be a composite number.If d^* is an arbitrary proper unitary divisor of n, then the inequalities*

$$\frac{\varphi(d^*)}{\sigma(d^*)} \le 1-\min_p\left\{\frac{1}{p}\left(1+\frac{\varphi(p)}{\sigma(p^{\mathrm{ord}_p n})}\right)\right\}$$

$$\frac{\varphi(d^*)}{\sigma(d^*)} \le \max_p\left\{\left(1-\frac{1}{p}\right)\left(1-\frac{1}{\sigma(p^{\mathrm{ord}_p n})}\right)\right\},$$

where p runs over all prime divisors of n, hold.

Proof. The assertion follows from (*) and (**) because of the relation:

$$\frac{\sigma\left(\frac{n}{d^*}\right)}{\sigma(n)} = \frac{1}{\sigma(d^*)},$$

which is true since d^* is unitary divisor of n.

Corollary 2. *Under the conditions of Corollary 1 the following two inequalities hold:*

$$\max_{d\in\mathbb{D}_n^{**}}\frac{\varphi(d)}{\sigma(d)} \le 1-\min_p\left\{\frac{1}{p}\left(1+\frac{\varphi(p)}{\sigma(p^{\mathrm{ord}_p n})}\right)\right\}$$

$$\max_{d\in\mathbb{D}_n^{**}}\frac{\varphi(d)}{\sigma(d)} \le \max_p\left\{\left(1-\frac{1}{p}\right)\left(1-\frac{1}{\sigma(p^{\mathrm{ord}_p n})}\right)\right\},$$

where p runs over all prime divisors of n.

3.1 Further applications of the results

Let $n > 1$ be a composite number and let us choose again: $f = \varphi,\ g = \sigma$. Since for a prime p we have:

$$\sigma(p) = p+1;\ \frac{p+1}{p-1} = 1+\frac{2}{p-1};\ \frac{p+1}{p} = 1+\frac{1}{p};$$
$$\frac{\varphi(p^{\alpha_p-1})}{\varphi(p^{\alpha_p})} = \begin{cases} \frac{1}{p-1} & \text{for } \alpha_p = 1, \\ \frac{1}{p} & \text{for } \alpha_p > 1 \end{cases},$$

where $\alpha_p \stackrel{\text{def}}{=} \mathrm{ord}_p n$, it is easy to observe that

$$\sigma(p)\frac{\varphi(p^{-1+\mathrm{ord}_p n})}{\varphi(p^{\mathrm{ord}_p n})} = \begin{cases} 1+\frac{2}{p-1} & \text{for } \mathrm{ord}_p n = 1, \\ 1+\frac{1}{p} & \text{for } \mathrm{ord}_p n > 1 \end{cases}$$

Let p' be the largest number among all prime divisors p of n, satisfying the condition $\mathrm{ord}_p n = 1$ and p'' be the largest number among all prime divisors p of n, satisfying the condition $\mathrm{ord}_p n > 1$. Then putting

$$\delta_1 \overset{\mathrm{def}}{=} \min\left(1+\frac{2}{p'-1}, 1+\frac{1}{p''}\right) \tag{15}$$

it is trivial to see that

$$\delta_1 = \min_p \left\{\sigma(p)\frac{\varphi(p^{-1+\mathrm{ord}_p n})}{\varphi(p^{\mathrm{ord}_p n})}\right\} \tag{16}$$

where p runs over all prime divisors of n.

Since the pair (φ,σ) has the property $\mathbb{S}$ (satisfying (ii)), (8) and (16) yield

$$\min_{d\in\mathbb{D}_n^*}\left\{\varphi(d)\sigma\left(\frac{n}{d}\right)\right\} = \delta_1\varphi(n) \tag{17}$$

Here we remind that $\max\limits_{d\in\mathbb{D}_n^*}\left\{\varphi(d)\sigma\left(\frac{n}{d}\right)\right\}$ was found with the help of Theorem 11 (see (*) and (**)).

Let ψ be the Dedekind's function. Since $\psi\in\mathbb{M}$ and for a prime p we have

$$\psi(p) = \sigma(p) = p+1,$$

using the fact that the pair (φ,ψ) has the property $\mathbb{S}$ (satisfying (ii)), we obtain that for a composite $n>1$ the representation

$$\min_{d\in\mathbb{D}_n^*}\left\{\varphi(d)\psi\left(\frac{n}{d}\right)\right\} = \delta_1\varphi(n) \tag{18}$$

holds.

Let d be a proper unitary divisor of n. Then $\frac{n}{d}$ is also a proper unitary divisor of n and we have

$$\varphi(d)\sigma\left(\frac{n}{d}\right) = \frac{\sigma\left(\frac{n}{d}\right)}{\varphi\left(\frac{n}{d}\right)} \tag{19}$$

Therefore, when d runs over all proper unitary divisors of n, so does $\frac{n}{d}$ and (17), (18) and (19) yield for every proper unitary divisor d of n the inequality:

$$\min\left(\frac{\sigma(d)}{\varphi(d)}, \frac{\psi(d)}{\varphi(d)}\right) \geq \delta_1$$

Using the facts that for a prime p :

$$\varphi(p) = p-1; \frac{p-1}{p+1} = 1+\frac{2}{p+1}; \ \frac{p-1}{p} = 1-\frac{1}{p};$$

$$\frac{\psi(p^{\mathrm{ord}_p n-1})}{\psi(p^{\mathrm{ord}_p n})} = \begin{cases} \frac{1}{p+1} & \text{for } \mathrm{ord}_p n = 1, \\ \frac{1}{p} & \text{for } \mathrm{ord}_p n > 1 \end{cases},$$

we find

$$\varphi(p)\frac{\psi(p^{-1+\mathrm{ord}_p n})}{\psi(p^{\mathrm{ord}_p n})} = \begin{cases} 1-\frac{2}{p+1} & \text{for } \mathrm{ord}_p n = 1, \\ 1-\frac{1}{p} & \text{for } \mathrm{ord}_p n > 1 \end{cases}$$

The above equality yields:

$$\delta_2 = \max_p \left\{ \varphi(p) \frac{\psi(p^{-1+\mathrm{ord}_p n})}{\psi(p^{\mathrm{ord}_p n})} \right\},$$

where p runs over all prime divisors of n and $\delta_2 \overset{\text{def}}{=} \max\left(1 - \frac{2}{p'+1}, 1 - \frac{1}{p''}\right)$. The last equality and (7) (for $f = \varphi, g = \psi$) yield:

$$\max_{d \in \mathbb{D}_n^*} \left\{ \varphi(d)\psi\left(\frac{n}{d}\right) \right\} = \delta_2 \psi(n)$$

The last equality and (18) prove the following **fourth main result** in the paper:

Theorem 12. *Let $n > 1$ be a composite number. Then it is fulfilled:*

$$\max_{d \in \mathbb{D}_n^*} \left\{ \varphi(d)\psi\left(\frac{n}{d}\right) \right\} = \delta_2 \psi(n)$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ \varphi(d)\psi\left(\frac{n}{d}\right) \right\} = \delta_1 \varphi(n)$$

Corollary 3. *For every proper unitary divisor d of n it is fulfilled*

$$\delta_1 \varphi(n) \le \varphi(d)\psi\left(\frac{n}{d}\right) \le \delta_2 \psi(n)$$

Moreover, there exist values of d for which the left and right inequalities become equalities.

Corollary 4. *Let $n > 1$ be a composite number and d be an arbitrary proper unitary divisor of n. Then the inequalities:*

$$\frac{\psi(d)}{\varphi(d)} \ge \delta_1$$

$$\frac{\varphi(d)}{\psi(d)} \le \delta_2$$

hold.

Remark 2. *We must note that if $n > 1$ is a composite number such that for every prime divisor p of n the condition $\mathrm{ord}_p n = 1$ is satisfied, then $\delta_1 = 1 + \frac{2}{p'-1}$ and $\delta_2 = 1 - \frac{2}{1+p'}$.*

Remark 3. *If $n > 1$ is a composite number such that for every prime divisor p of n the condition $\mathrm{ord}_p n > 1$ is satisfied, then $\delta_1 = 1 + \frac{1}{p''}$ and $\delta_2 = 1 - \frac{1}{p''}$.*

From the above remarks it follows:

Corollary 5. *Let $n > 1$ be a composite square-free number. Then for every proper divisor d of n it is fulfilled*

$$\left(1 + \frac{2}{p-1}\right) \varphi(n) \le \varphi(d)\psi\left(\frac{n}{d}\right) \le \left(1 - \frac{2}{p+1}\right) \psi(n)$$

where p is the largest prime divisor of n. Moreover, there exist values of d for which the left and right inequalities become equalities.

Corollary 6. *Let $n > 1$ be a composite number such that if q is prime divisor of n, q^2 is also a divisor of n. Then for every proper divisor d of n it is fulfilled*

$$\left(1+\frac{1}{p}\right)\varphi(n) \le \varphi(d)\psi\left(\frac{n}{d}\right) \le \left(1-\frac{1}{p}\right)\psi(n)$$

where p is the largest prime divisor of n. Moreover, there exist values of d for which the left and right inequalities become equalities.

One may conjecture:

"For every composite $n > 1$ $\max\limits_{d\in\mathbb{D}_n^*}\varphi(d)\sigma\left(\frac{n}{d}\right)$ is reached for $d = p^*$, where p^* is the maximal prime divisor of n."

Indeed this is true for infinitely many n, but unfortunately the conjecture is untrue for infinitely many values of n too. Below we construct one possible counter-example.

Let $p > 3$ be prime and q is a prime number for which $p < q \le 2p-3$. Such prime q always exists according to Bertrand's postulate (see e.g. [3]): "For $p > 3$, there always exists a prime number q between p and $2p-2$ (i.e. $q \in (p, 2p-2)$)." In this form Bertrand's postulate follows immediately from Nagura's result (see [4]): "For every $p \ge 25$, there always exists a prime $q \in (p, \frac{6}{5}p)$." It is clear that when p runs over all primes greater than 3, we obtain infinitely many pairs (p, q) like the described above. For every such pair (p, q) we put $n = p^2 q$. Then all elements of the set D_n^* ordered by increasing magnitude are: p, q, p^2, pq. Let $q = p + x$. Then it is easy to see that the inequality

$$\varphi(p)\sigma\left(\frac{n}{p}\right) > \varphi(q)\sigma\left(\frac{n}{q}\right) \tag{20}$$

is equivalent to

$$x < \frac{p^2-p}{p+2}.$$

And since we have, $x \le p-3$, and $p-3 < \frac{p^2-p}{p+2}$, then (20) holds. Also a direct check shows that the inequalities:

$$\varphi(p)\sigma\left(\frac{n}{p}\right) > \varphi(p^2)\sigma\left(\frac{n}{p^2}\right);\ \varphi(p)\sigma\left(\frac{n}{p}\right) > \varphi(pq)\sigma\left(\frac{n}{pq}\right)$$

hold too. Therefore, $\max\limits_{d\in\mathbb{D}_n^*}\varphi(d)\sigma\left(\frac{n}{d}\right)$ is reached for $d = p$, i.e. for the smallest prime divisor of n, while in the considered case $p^* = q$.

It is easy to observe that $\min\limits_{d\in\mathbb{D}_n^*}\varphi(d)\sigma\left(\frac{n}{d}\right)$ in each one of the above infinitely many counter examples is reached for $d = \frac{n}{p}$ and based on that the following conjecture can be formulated:

"For every composite $n > 1$ if $\max\limits_{d\in\mathbb{D}_n^*}\varphi(d)\sigma\left(\frac{n}{d}\right)$ is reached for $d = \hat{d}$, then $\min\limits_{d\in\mathbb{D}_n^*}\varphi(d)\sigma\left(\frac{n}{d}\right)$ is reached at $d = \frac{n}{\hat{d}}$."

Unfortunately, this conjecture is also untrue. One may verify that a counter-example is every n given by:

$$n = p^2(2p-1),$$

where the numbers $p > 3$ and $2p-1$ are simultaneously prime.

Let us consider the classical arithmetic function τ given by:

$$\tau(m) = \sum_{d|m} 1$$

where d runs over all divisors of m. Then $\tau(m)$ coincides with the number of all divisors of m. Moreover, $\tau \in \mathbb{M}$. It can be checked directly that the pair (τ, σ) has the property $\mathbb{S}$ (satisfying (ii)). Therefore, we may substitute $f = \tau, g = \sigma$ and applying (7) and (8) obtain:

$$\max_{d\in\mathbb{D}_n^*} \left\{\tau(d)\sigma\left(\frac{n}{d}\right)\right\} = \sigma(n) \max_p \left\{\tau(p) \frac{\sigma\left(p^{-1+\mathrm{ord}_p n}\right)}{\sigma\left(p^{ord_p n}\right)}\right\} \tag{21}$$

$$\min_{d\in\mathbb{D}_n^*} \left\{\tau(d)\sigma\left(\frac{n}{d}\right)\right\} = \tau(n) \min_p \left\{\sigma(p) \frac{\tau\left(p^{-1+\mathrm{ord}_p n}\right)}{\tau\left(p^{\mathrm{ord}_p n}\right)}\right\} \tag{22}$$

where p runs over all prime divisors of n.

Since $\tau(p) = 2$ and

$$\frac{\sigma\left(p^{-1+\mathrm{ord}_p n}\right)}{\sigma\left(p^{\mathrm{ord}_p n}\right)} = \frac{p^{\mathrm{ord}_p n} - 1}{p^{1+\mathrm{ord}_p n} - 1} = \frac{1}{p}\left(1 - \frac{p-1}{p^{1+\mathrm{ord}_p n} - 1}\right) = \frac{1}{p}\left(1 - \frac{1}{\sigma\left(p^{\mathrm{ord}_p n}\right)}\right) \tag{23}$$

we may rewrite (21) in the form

$$\max_{d\in\mathbb{D}_n^*} \left\{\tau(d)\sigma\left(\frac{n}{d}\right)\right\} = 2\sigma(n) \max_p \left\{\frac{1}{p}\left(1 - \frac{1}{\sigma\left(p^{\mathrm{ord}_p n}\right)}\right)\right\} \tag{24}$$

where p runs over all prime divisors of n.

Since $\sigma(p) = p + 1$ and

$$\tau\left(p^{-1+\mathrm{ord}_p n}\right) = \mathrm{ord}_p n;\ \tau\left(p^{\mathrm{ord}_p n}\right) = 1 + \mathrm{ord}_p n$$

we may rewrite (22) in the form

$$\min_{d\in\mathbb{D}_n^*} \left\{\tau(d)\sigma\left(\frac{n}{d}\right)\right\} = \tau(n) \min_p \left\{(p+1)\left(1 - \frac{1}{1 + \mathrm{ord}_p n}\right)\right\} \tag{25}$$

where p runs over all prime divisors of n.

Then (24) and (25) yield the **fifth main result** of the paper:

Theorem 13. *Let $n > 1$ be a composite number. Then the relations:*

$$\max_{d\in\mathbb{D}_n^*} \left\{\tau(d)\sigma\left(\frac{n}{d}\right)\right\} = 2\sigma(n) \max_p \left\{\frac{1}{p}\left(1 - \frac{1}{\sigma\left(p^{ord_p n}\right)}\right)\right\}$$

$$\min_{d\in\mathbb{D}_n^*} \left\{\tau(d)\sigma\left(\frac{n}{d}\right)\right\} = \tau(n) \min_p \left\{(p+1)\left(1 - \frac{1}{1 + ord_p n}\right)\right\},$$

where p runs over all prime divisors of n, hold.

Corollary 7. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where $p_1, p_2 \in \mathbb{P}, p_1 \neq p_2$ and $\alpha_1, \alpha_2 \in \mathbb{N}$. Then:*

$$\max_{d \in \mathbb{D}_n^*} \left\{ \tau(d) \sigma\left(\frac{n}{d}\right) \right\} = 2\sigma(n) \max\left(\frac{1}{p_1}\left(1 - \frac{1}{\sigma\left(p_1^{\alpha_1}\right)}\right), \frac{1}{p_2}\left(1 - \frac{1}{\sigma\left(p_2^{\alpha_2}\right)}\right) \right);$$

$$\min_{d \in \mathbb{D}_n^*} \left\{ \tau(d) \sigma\left(\frac{n}{d}\right) \right\} = \tau(n) \min\left((p_1 + 1)\left(1 - \frac{1}{\alpha_1 + 1}\right), (p_2 + 1)\left(1 - \frac{1}{\alpha_2 + 1}\right) \right)$$

It is well known that $2^{\omega(m)}$ is multiplicative function. Since $2^{\omega(m)} \in \mathbb{M}$, it is easy to verify that the pair $(2^{\omega(m)}, \sigma)$ has the property $\mathbb{S}$ (satisfying (ii)). Let $n > 1$ be a composite number. Then applying (7) with $f = 2^{\omega(m)}$ and $g = \sigma$ we obtain:

$$\max_{d \in \mathbb{D}_n^*} \left\{ 2^{\omega(d)} \sigma\left(\frac{n}{d}\right) \right\} = \sigma(n) \max_p \left\{ 2^{\omega(p)} \frac{\sigma\left(p^{-1+\mathrm{ord}_p n}\right)}{\sigma\left(p^{\mathrm{ord}_p n}\right)} \right\},$$

where p runs over all prime divisors of n. Hence:

$$\max_{d \in \mathbb{D}_n^*} \left\{ 2^{\omega(d)} \sigma\left(\frac{n}{d}\right) \right\} = 2\sigma(n) \max_p \left\{ \frac{\sigma\left(p^{-1+\mathrm{ord}_p n}\right)}{\sigma\left(p^{\mathrm{ord}_p n}\right)} \right\} \tag{26}$$

(where p runs over all prime divisors of n), since $\omega(p) = 1$.

Since $\tau(p) = 2$ (21) yields

$$\max_{d \in \mathbb{D}_n^*} \left\{ \tau(d) \sigma\left(\frac{n}{d}\right) \right\} = 2\sigma(n) \max_p \left\{ \frac{\sigma\left(p^{-1+\mathrm{ord}_p n}\right)}{\sigma\left(p^{\mathrm{ord}_p n}\right)} \right\},$$

where p runs over all prime divisors of n. Comparing the last equality with (26) we conclude that:

$$\max_{d \in \mathbb{D}_n^*} \left\{ \tau(d) \sigma\left(\frac{n}{d}\right) \right\} = \max_{d \in \mathbb{D}_n^*} \left\{ 2^{\omega(d)} \sigma\left(\frac{n}{d}\right) \right\} \tag{27}$$

Also, from (23) and (26) we obtain

$$\max_{d \in \mathbb{D}_n^*} \left\{ 2^{\omega(d)} \sigma\left(\frac{n}{d}\right) \right\} = 2\sigma(n) \max_p \left\{ \frac{1}{p}\left(1 - \frac{1}{\sigma\left(p^{\mathrm{ord}_p n}\right)}\right) \right\}, \tag{28}$$

where p runs over all prime divisors of n.

Putting in (8) $f = 2^{\omega(m)}$ and $g = \sigma$ we obtain

$$\min_{d \in \mathbb{D}_n^*} \left\{ 2^{\omega(d)} \sigma\left(\frac{n}{d}\right) \right\} = 2^{\omega(n)} \min_p \left\{ (p+1) \frac{2^{\omega\left(p^{-1+\mathrm{ord}_p n}\right)}}{2^{\omega\left(p^{\mathrm{ord}_p n}\right)}} \right\},$$

where p runs over all prime divisors of n. Hence:

$$\min_{d \in \mathbb{D}_n^*} \left\{ 2^{\omega(d)} \sigma\left(\frac{n}{d}\right) \right\} = 2^{\omega(n)} \min_p \left\{ (p+1) \frac{2^{\nu_n(p)}}{2} \right\} \tag{29}$$

(where p runs over all prime divisors of n), where

$$\nu_n(p) \overset{\text{def}}{=} \begin{cases} 0 & \text{for } \mathrm{ord}_p n = 1, \\ 1 & \text{for } \mathrm{ord}_p n > 1 \end{cases}$$

Let

$$p^* = \{p : p \in \mathbb{P}, p|n \ \& \ \mathrm{ord}_p n > 1\}$$

$$p^{**} = \{p : p \in \mathbb{P}, p|n \ \& \ \mathrm{ord}_p n = 1\}$$

Then (27), (28) and (29) prove the **sixth main result** of this paper:

Theorem 14. *Let $n > 1$ be a composite number. Then the following relations hold:*

$$\max_{d \in \mathbb{D}_n^*} \left\{ 2^{\omega(d)} \sigma \left(\frac{n}{d} \right) \right\} = 2\sigma(n) \max_p \left\{ \frac{1}{p} \left(1 - \frac{1}{\sigma\left(p^{ord_p n}\right)} \right) \right\}$$

(where p runs over all prime divisors of n);

$$\min_{d \in \mathbb{D}_n^*} \left\{ 2^{\omega(d)} \sigma \left(\frac{n}{d} \right) \right\} = 2^{\omega(n)} \min \left(1 + p^* \ , \frac{1 + p^{**}}{2} \right);$$

$$\max_{d \in \mathbb{D}_n^*} \left\{ \tau(d) \sigma \left(\frac{n}{d} \right) \right\} = \max_{d \in \mathbb{D}_n^*} \left\{ 2^{\omega(d)} \sigma \left(\frac{n}{d} \right) \right\}.$$

We conclude this paper with a demonstration of the effectiveness of the established results by providing a simple example.

Let $n = 2^{999}3^{9999}$. Then the set D_n^* has exactly 9999998 in number elements. That means that the number of all proper divisors of n is 9999998. Now it is clear that to find $\min_{d \in \mathbb{D}_n^*} \left\{ \tau(d) \sigma \left(\frac{n}{d} \right) \right\}$ is not a simple problem. But after using the second formula of the Corollary 7, we find immediately that

$$\min_{d \in \mathbb{D}_n^*} \left\{ \tau(d) \sigma \left(\frac{n}{d} \right) \right\} = 10^7 \left(\min \left(3 \left(1 - \frac{1}{1000} \right), 4 \left(1 - \frac{1}{10000} \right) \right) \right) = 29970000$$

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