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On some application of the spectral properties of the matrices [1]

Aleksander Grytczuk and Izabela Kurzydło

Faculty of Mathematics, Computer Science and Econometrics,
University of Zielona Góra,
65-516 Zielona Góra, Poland
emails: $\{A.Grytczuk, I.Kurzydlo\}@wmie.uz.zgora.pl$

Abstract. In the paper [14] A. Schinzel and H. Zassenhaus posed the following conjecture:

If $\alpha \neq 0$ is an algebraic integer of degree n which is not a root of unity, then there exists a constant $c > 0$ such that

$$|\overline{\alpha}| \geq 1 + \frac{c}{n}, \text{ where } |\overline{\alpha}| = \max_{1 \leq i \leq n} |\alpha_i| ,$$

where $\alpha = \alpha_1$, and $\alpha_2, ..., \alpha_n$ are the conjugates of α.

In this paper we give some information concerning this conjecture. In the proofs of the theorems we use some spectral properties of matrices.

Keywords. Conjecture of A. Schinzel and H. Zassenhaus, Spectral properties of matrices

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1 Introduction

In the paper [14] A. Schinzel and H. Zassenhaus posed the following conjecture:

If $\alpha \neq 0$ is an algebraic integer of degree n which is not a root of unity, then there exists a constant $c > 0$ such that

$$|\overline{\alpha}| \quad \geq \quad 1 + \frac{c}{n}, \text{ where } |\overline{\alpha}| = \max_{1 \leq i \leq n} |\alpha_i| ,$$

where $\alpha = \alpha_1$, and $\alpha_2, ..., \alpha_n$ are the conjugates of α.

In this paper we give some information concerning this conjecture.

This conjecture is strictly connected with another open Lehmer's conjecture. Lehmer asked in 1933 [9] whether there exists a constant $c\prime > 0$ such that for all algebraic integers $\alpha \neq 0$ which are not roots of unity we have

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$$M(\alpha) = \prod_{i=1}^{n} \max(1, |\overline{\alpha}|) \geq 1 + c'.$$

Studies connected with the conjecture of Schinzel and Zassenhaus effected Blanksby and Montgomery [1], Stewart [15], Dobrowolski [4], Boyd [2]. Some results relating to this conjecture have been described by W. Narkiewicz in [12]. In 1979 Dobrowolski in [5] proved that

$$|\overline{\alpha}| > 1 + (2 - \varepsilon) \left(\frac{\log \log n}{\log n} \right)^3 \frac{1}{n} \quad \text{for } n > n_0(\varepsilon). \tag{1}$$

Cantor and Straus [3] improved the constant $2-\varepsilon$ in (1) to $4-\varepsilon$. Louboutin [10] replaced $4-\varepsilon$ by $\frac{9}{2}-\varepsilon$. In 1993 Dubickas in [6] improved this result to $\frac{64}{\pi^2}-\varepsilon$. In [11] Matveev showed that if α is the algebraic integer conjugated to $\alpha^{-1}$with the degree $n = 2m \geq 6$, p is the least prime number exceeding m and $|\overline{\alpha}| < \exp\left(\frac{\ln(p-1)}{pm}\right)$,then α is a root of unity.

In 2007 Rhin and Wu [13] gave the estimation of the minimum of $|\overline{\alpha}|$ for all algebraic integers of degree $n \leq 28$ which are not roots of unity.

2 Results

In the proofs we use the following Lemmas:

Lemma 1 ([7]). *Let $A \in M_n(\mathbb{Z})$, where $M_n(\mathbb{Z})$ denotes the set of $n \times n$ matrices over $\mathbb{Z}$ and let A be the non-singular matrix.*

The necessary and sufficient condition for λ_i, where λ_i are characteristic roots of A, $i = 1, ..., n$, to be roots of unity is

$$|\lambda_i| = 1$$

for $i = 1, ..., n$.

Lemma 2 ([7]). *Let A be an $n \times n$ complex matrix with $|\det A| > 1$ and let*

$$\left|\overline{\lambda}\right| = \max_{1 \leq\ i\ \leq n} |\lambda_i|,$$

where λ_i are the eigenvalues of A for $i = 1, ..., n$, then

$$\left|\overline{\lambda}\right| \geq 1 + \frac{\log|\det A|}{n}.$$

Theorem 1. *If $\alpha \neq 0$ is an algebraic integer of degree n which is not a root of unity, and $I(\alpha) := |\ \alpha_{1...}\alpha_n| \neq 1$, where $\alpha = \alpha_1$, and $\alpha_2, ..., \alpha_n$ are the conjugates of α, then there exists a constant $c = \log 2$ such that*

$$|\overline{\alpha}| \geq 1 + \frac{c}{n}, \ \textit{where}\ |\overline{\alpha}| = \max_{1 \leq\ i\ \leq n} |\alpha_i|\ .$$

Proof. Consider the following matrix:

$$A_f = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ \\ 0 & 0 & 0 & . & . & . & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & . & . & . & -a_2 & -a_1 \end{pmatrix}, \tag{2}$$

where $a_j \in Z;\ 1 \leq j \leq n.$

The matrix A_f given in (2) is well-know as associated matrix with polynomial $f(\alpha)$ of the form

$$f(\alpha) = \det(\alpha I - A_f) = \alpha^n + a_1\alpha^{n-1} + ... + a_{n-1}\alpha + a_n \tag{3}$$

(see [8]), where the characteristic roots α_i of A_f, $i = 1, ..., n$, are algebraic integers of the degree n.

From the assumption and Lemma 1 we get $|\alpha| = |\alpha_1| \neq 1$.

Since $A_f \in M_n(\mathbb{Z})$, then det A_f is a integer. Hence and from the assumption that $I(\alpha) = |\ \alpha_{1...}\alpha_n| \neq 1$ we obtain

$$|\det A_f| = |\alpha_{1...}\alpha_n| > 1.$$

By Lemma 2 it follows that

$$|\overline{\alpha}| \geq 1 + \frac{\log\ \ |\det A_f|}{n}.$$

Since $|\det A_f| > 1$ and $\det A_f \in \mathbb{Z}$, then

$$\log|\det A_f| \geq \log 2.$$

Therefore

$$|\overline{\alpha}| \geq 1 + \frac{\log\ 2}{n}.$$

The proof of Theorem 1 is complete. □

It is known that the coefficient a_n of the polynomial of the form (3) is given by formula

$$a_n = (-1)^n\alpha_1\alpha_2 ...\ \alpha_n. \tag{4}$$

Since by (4) it follows that $\alpha_1\alpha_2 ...\ \alpha_n \neq 0$ and moreover we have $\alpha_1\alpha_2 ...\ \alpha_n \in \mathbb{Z}$, then it remains to consider the case when

$$I(\alpha) = |\alpha_{1 \dots} \alpha_n| \; = 1.$$

Then $|\overline{\alpha}| > 1$.

We prove the following theorem:

Theorem 2. *Let α be an algebraic integer of degree $n > \dfrac{1-\varepsilon}{\varepsilon} \log 2$ such that $|\alpha| > 1$ and let $I(\alpha) = |\alpha_{1 \dots} \alpha_n| = 1$, where $\alpha = \alpha_1$ and $\alpha_2, ..., \alpha_n$ are the conjugates of α. Then it holds the following inequality:*

$$|\overline{\alpha}| > 1 + \frac{\log\ 2}{n}, \; \textit{where } |\overline{\alpha}| = \max_{1 \leq\ i\ \leq n} |\alpha_i| \; .$$

Proof. Assume that

$$|\alpha| = |\overline{\alpha}| = \max_{1 \leq\ i\ \leq n} |\alpha_i|.$$

We can suppose of course that

$$|\alpha_2 \alpha_3 ... \ \alpha_n| < 1. \tag{5}$$

If it held the opposite inequality

$$|\alpha_2 \alpha_3 ... \ \alpha_n| \geqslant 1, \tag{6}$$

then from (6) and from the fact that $|\alpha| > 1$, we would have

$$I(\alpha) = |\alpha \alpha_{2 \dots} \alpha_n| \; > 1. \tag{7}$$

This case is solved in Theorem 1.

By (5) it follows so, that for some $\varepsilon \in (0, 1)$ we have

$$|\alpha_2 \alpha_3 ... \ \alpha_n| = 1 - \varepsilon. \tag{8}$$

From other hand, since $I(\alpha) = |\alpha \alpha_{2 \dots} \alpha_n| = 1$, we obtain

$$|\overline{\alpha}| = \frac{1}{|\alpha_2 \alpha_3 ... \ \alpha_n|}. \tag{9}$$

From (8) and (9) we get

$$|\overline{\alpha}| = \frac{1}{1 - \varepsilon}. \tag{10}$$

By the assumption that $n > \dfrac{1-\varepsilon}{\varepsilon} \log 2$; $\varepsilon \in (0, 1)$ it follows that

$$\begin{aligned} n\varepsilon &> (1-\varepsilon)\log 2, \\ n\frac{\varepsilon}{1-\varepsilon} &> \log 2, \end{aligned}$$

$$\frac{\varepsilon}{1-\varepsilon} > \frac{\log 2}{n}. \tag{11}$$

Since $\dfrac{\varepsilon}{1-\varepsilon} = \dfrac{1-1+\varepsilon}{1-\varepsilon}$, then from (11) we have

$$\frac{1-1+\varepsilon}{1-\varepsilon} > \frac{\log 2}{n}. \tag{12}$$

From (12) we get

$$\frac{1}{1-\varepsilon} - \frac{1-\varepsilon}{1-\varepsilon} > \frac{\log 2}{n}, \tag{13}$$

therefore

$$\frac{1}{1-\varepsilon} > 1 + \frac{\log 2}{n}. \tag{14}$$

From (10) and (14) we obtain

$$|\overline{\alpha}| > 1 + \frac{\log\ 2}{n},$$

and the proof of Theorem 2 is finished. □

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