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SOME RESULTS ON MULTIPLICATIVE FUNCTIONS

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Abstract: In the present paper some new results, concerning multiplicative functions with strictly positive values, are obtained. In particular, it is shown that if an ordered pair of such functions (f, g) has a certain property (called in the paper $\mathbb{S}$), then for every fixed positive integer n, the minimal and the maximal elements of the set $\{f(d)g\left(\frac{n}{d}\right)$: d runs over all divisors of $n\}$ are obtained at least for some unitary divisors of n. For these divisors if the maximum of $f(d)g(\frac{n}{d})$ is reached for d^*, then the minimum is reached for $\frac{n}{d^*}$ and vice versa (the main results here are Theorems 1-4). The same ivestigation is made, but when d runs over the set of all divisors of n different than 1 and n (the main result here is Theorem 5). Also corollaries of the mentioned results are obtained and some particular cases are considered.
Keywords: Multiplicative functions, Divisors, Unitary divisors, Non-unitary divisors

1. Introduction

1.1. Used Denotations

$\mathbb{Z}^+$ - the set of all non-negative integers; $\mathbb{N}$ - the set of all positive integers; $\mathbb{P}$ - the set of all prime numbers; for a given $n \in \mathbb{N}$: $\mathbb{D}_n$ denotes the set of all divisors of n (including 1 and n); $\mathbb{D}'_n$ - the set of all unitary divisors of n (see [1]), i.e. the set of all $d' \in \mathbb{D}_n$ such that $\gcd(d', \frac{n}{d'}) = 1$; $\mathbb{D}''_n$ - the set of all non-unitary divisors of n, i.e. the set of all $d'' \in \mathbb{D}_n$ such that $\gcd(d'', \frac{n}{d''}) > 1$; D^*_n- the set of all divisors of n different than 1 and n; $\mathbb{M}$ - the set of all multiplicative functions with strictly positive values; for $f, g \in \mathbb{M}$ we introduce: $L_n(f,g) \stackrel{\text{def}}{=} \{d^* : d^* \in \mathbb{D}_n \ \& \ f(d^*)g\left(\frac{n}{d^*}\right) = \max\limits_{d\in\mathbb{D}_n} f(d)g\left(\frac{n}{d}\right)\}$; $l_n(f,g) \stackrel{\text{def}}{=} \{d^* : d^* \in \mathbb{D}_n \ \& \ f(d^*)g\left(\frac{n}{d^*}\right) = \min\limits_{d\in\mathbb{D}_n} f(d)g\left(\frac{n}{d}\right)\}$; $L'_n(f,g) \stackrel{\text{def}}{=} \{d^* : d^* \in \mathbb{D}'_n \ \& \ f(d^*)g\left(\frac{n}{d^*}\right) = \max\limits_{d\in\mathbb{D}'_n} f(d)g\left(\frac{n}{d}\right)\}$; $l'_n(f,g) \stackrel{\text{def}}{=} \{d^* : d^* \in \mathbb{D}'_n \ \& \ f(d^*)g\left(\frac{n}{d^*}\right) = \min\limits_{d\in\mathbb{D}'_n} f(d)g\left(\frac{n}{d}\right)\}$; $L''_n(f,g) \stackrel{\text{def}}{=} \{d^* : d^* \in \mathbb{D}''_n \ \& \ f(d^*)g\left(\frac{n}{d^*}\right) = \max\limits_{d\in\mathbb{D}''_n} f(d)g\left(\frac{n}{d}\right)\}$; $l''_n(f,g) \stackrel{\text{def}}{=} \{d^* : d^* \in \mathbb{D}''_n \ \& \ f(d^*)g\left(\frac{n}{d^*}\right) = \min\limits_{d\in\mathbb{D}''_n} f(d)g\left(\frac{n}{d}\right)\}$.

1.2. Observations regarding the above denotations

It is clear that:

$$\mathbb{D}_n = \mathbb{D}'_n \cup \mathbb{D}''_n; \quad \mathbb{D}'_n \cap \mathbb{D}''_n = \emptyset$$

and $\mathbb{D}'_n \neq \emptyset$ since $1 \in \mathbb{D}'_n$ and $n \in \mathbb{D}'_n$, while $\mathbb{D}''_n = \emptyset$ at least when $n \in \mathbb{P}$.

Also, it is clear that $L_n(f,g), l_n(f,g), L'_n(f,g), l'_n(f,g)$ always exist and are non-empty sets, while $L''_n(f,g)$ and $l''_n(f,g)$ are non-empty sets only when $\mathbb{D}''_n \neq \emptyset$. The following relations are obvious:

$$L'_n(f,g) \cap l'_n(f,g) = \emptyset;$$

$$L''_n(f,g) \cap l''_n(f,g) = \emptyset;$$

$$L_n(f,g) \subseteq L'_n(f,g) \cup L''_n(f,g);$$

$$l_n(f,g) \subseteq l'_n(f,g) \cup l''_n(f,g).$$

2. Main results

The set of all multiplicative functions is well studied. We recommend the following book [2] to the interested reader, wherein many properties and results are collected. Below we shall study some new properties of a class of multiplicative functions. First we need the following:

Definition. *Let $f, g \in \mathbb{M}$. We say that the ordered pair (f,g) has the property $\mathbb{S}$ when one of the following two cases is fulfilled:*

(i) $\forall p \in \mathbb{P}$ & $\forall m \in \mathbb{Z}^+$

$$H^{f,g}_{p,m}(k) \stackrel{\text{def}}{=} f(p^k)g(p^{m-k}) \tag{1}$$

is an increasing function (not necessarily strictly) with respect to $k \in [0,m] \cap \mathbb{Z}^+$

(ii) $\forall p \in \mathbb{P}$ & $\forall m \in \mathbb{Z}^+$ *the function $H^{f,g}_{p,m}$ from (1) is a decreasing function (not necessarily strictly) with respect to $k \in [0,m] \cap \mathbb{Z}^+$*

Our first main result is:

Theorem 1. *Let $n \in \mathbb{N}$ is such that $\mathbb{D}''_n \neq \emptyset$ and $f, g \in \mathbb{M}$ are such that the ordered pair (f,g) has the property $\mathbb{S}$. Then:*

$$\max_{d \in \mathbb{D}''_n} f(d)g\left(\frac{n}{d}\right) \leq \max_{d \in \mathbb{D}'_n} f(d)g\left(\frac{n}{d}\right) \tag{$*_1$}$$

$$\min_{d \in \mathbb{D}'_n} f(d)g\left(\frac{n}{d}\right) \leq \min_{d \in \mathbb{D}''_n} f(d)g\left(\frac{n}{d}\right) \tag{$*_2$}$$

Proof. Let $d'' \in \mathbb{D}''_n$ and $\gcd(d'', \frac{n}{d''}) = r$. Then $r > 1$ and r admits the cannonical factorization in primes of the form:

$$r = \prod_{i=1}^{s} p_i^{\alpha_i} \tag{2}$$

where $s \in \mathbb{N}$, $\alpha_i \in \mathbb{Z}^+, i = 1, \ldots, s$ and $\sum_{i=1}^{s} \alpha_i > 0$. Since r is a divisor of d'', d'' admits the representation:

$$d'' = Q \prod_{i=1}^{s} p_i^{\beta_i} \tag{3}$$

where $\beta_i \geq \alpha_i$ and $Q \geq 1$ is coprime with each of $p_i, i =, 1, \ldots, s$. Let

$$P = \prod_{i=1}^{s} p_i^{\beta_i} \tag{4}$$

Then

$$d'' = PQ \tag{5}$$

with $\gcd(P, Q) = 1$. Since $d'' \in \mathbb{D}''_n$, then d'' is a divisor of n. Hence:

$$n = T. \prod_{i=1}^{s} p_i^{m_i} \tag{6}$$

where $m_i \geq \beta_i$ and $T \geq 1$ is coprime with each of $p_i, i = 1, \ldots, s$. From (3) and (6) it follows:

$$\frac{n}{d''} = \frac{T}{Q} \prod_{i=1}^{s} p_i^{m_i - \beta_i} \tag{7}$$

where $\frac{T}{Q} \in \mathbb{N}$ is coprime with each of $p_i, i = 1, \ldots, s$.

The element d'' generates the following two elements of $\mathbb{D}_n$ given by:

$$d = \frac{d''}{r} \tag{a_1}$$

$$d = rd'' \tag{a_2}$$

Since $f \in \mathbb{M}$ and (2) and (3) are valid, for the case (a_1) we obtain:

$$f(d) = f\left(\frac{d''}{r}\right) = f(Q) \prod_{i=1}^{s} p_i^{\beta_i - \alpha_i} = f(Q) \prod_{i=1}^{s} f(p_i^{\beta_i - \alpha_i})$$

From (5) we find:

$$f(Q) = \frac{f(d'')}{f(P)} \tag{8}$$

(with P from (4)) and subsitute (8) in the previous equality. Thus, using that $f \in \mathbb{M}$ we obtain:

$$f(d) = f(d'') \prod_{i=1}^{s} \frac{f(p_i^{\beta_i - \alpha_i})}{f(p_i^{\beta_i})} \tag{9}$$

In the same way $g \in \mathbb{M}$, (2), (6) and (7) yield:

$$g\left(\frac{n}{d}\right) = g\left(\frac{nr}{d''}\right) = g\left(\frac{T}{Q}\prod_{i=1}^{s} p_i^{m_i+\alpha_i-\beta_i}\right) = g\left(\frac{T}{Q}\right)\prod_{i=1}^{s} g\left(p_i^{m_i-(\beta_i-\alpha_i)}\right)$$

From (7) we find

$$g\left(\frac{T}{Q}\right) = \frac{g\left(\frac{n}{d''}\right)}{\prod_{i=1}^{s} g\left(p_i^{m_i-\beta_i}\right)} \tag{10}$$

and putting this in the previous equality obtain:

$$g\left(\frac{n}{d}\right) = g\left(\frac{n}{d''}\right)\prod_{i=1}^{s}\frac{g\left(p_i^{m_i-(\beta_i-\alpha_i)}\right)}{g\left(p_i^{m_i-\beta_i}\right)} \tag{11}$$

Now (9) and (11) yield:

$$f(d)g\left(\frac{n}{d}\right) = \xi.f(d'')g\left(\frac{n}{d''}\right) \tag{12}$$

where

$$\xi \stackrel{\text{def}}{=} \prod_{i=1}^{s}\xi_i \tag{13}$$

and

$$\xi_i \stackrel{\text{def}}{=} \frac{H^{f,g}_{p_i,m_i}(\beta_i-\alpha_i)}{H^{f,g}_{p_i,m_i}(\beta_i)}, i = 1, \ldots, s. \tag{14}$$

The case (a_2), because of (2), (3) and $f \in \mathbb{M}$, yields:

$$f(d) = f(rd'') = f\left(Q\prod_{i=1}^{s} p_i^{\alpha_i+\beta_i}\right) = f(Q)\prod_{i=1}^{s} f(p_i^{\alpha_i+\beta_i})$$

Using (8) and the above equality we obtain:

$$f(d) = f(d'')\prod_{i=1}^{s}\frac{f(p_i^{\alpha_i+\beta_i})}{f(p_i^{\beta_i})} \tag{15}$$

In the same way $g \in \mathbb{M}$, (2),(6) and (7) yield:

$$g\left(\frac{n}{d}\right) = g\left(\frac{n}{rd''}\right) = g\left(\frac{T}{Q}\prod_{i=1}^{s}\frac{p_i^{m_i}}{p_i^{\alpha_i+\beta_i}}\right) = g\left(\frac{T}{Q}\prod_{i=1}^{s} p_i^{m_i-(\alpha_i+\beta_i)}\right) = g\left(\frac{T}{Q}\right)\prod_{i=1}^{s} g\left(p_i^{m_i-(\alpha_i+\beta_i)}\right)$$

After using (10), the above equality yields:

$$g\left(\frac{n}{d}\right) = g\left(\frac{n}{d''}\right)\prod_{i=1}^{s}\frac{g\left(p_i^{m_i-(\alpha_i+\beta_i)}\right)}{g(p_i^{m_i-\beta_i})} \tag{16}$$

From (15) and (16) we obtain

$$f(d)g\left(\frac{n}{d}\right) = \eta f(d'')g\left(\frac{n}{d''}\right) \tag{17}$$

where

$$\eta \stackrel{\text{def}}{=} \prod_{i=1}^{s} \eta_i \tag{18}$$

and

$$\eta_i \stackrel{\text{def}}{=} \frac{H^{f,g}_{p_i,m_i}(\alpha_i+\beta_i)}{H^{f,g}_{p_i,m_i}(\beta_i)}, i = 1,\ldots,s. \tag{19}$$

Further we shall consider the following four possibilites:

$$(a_1) \text{ is valid \& } \textit{(i)} \text{ holds} \tag{a_{11}}$$

$$(a_1) \text{ is valid \& } \textit{(ii)} \text{ holds} \tag{a_{12}}$$

$$(a_2) \text{ is valid \& } \textit{(i)} \text{ holds} \tag{a_{21}}$$

$$(a_2) \text{ is valid \& } \textit{(ii)} \text{ holds} \tag{a_{22}}$$

For (a_{11}) we have $\xi_i \le 1,\ i = 1,\ldots,s$ and therefore $\xi \le 1$ (see (12), (13) and (14)). Hence:

$$f(d)g\left(\frac{n}{d}\right) \le f(d'')g\left(\frac{n}{d''}\right) \tag{20}$$

For (a_{12}) we have $\xi_i \ge 1,\ i = 1,\ldots,s$ and therefore $\xi \ge 1$. Hence:

$$f(d)g\left(\frac{n}{d}\right) \ge f(d'')g\left(\frac{n}{d''}\right) \tag{21}$$

For (a_{21}) we have $\eta_i \ge 1,\ i = 1,\ldots,s$ and therefore $\eta \ge 1$ (see (17), (18) and (19)). Hence (21) holds.

For (a_{22}) we have $\eta_i \le 1,\ i = 1,\ldots,s$ and therefore $\eta \le 1$. Hence (20) holds.

Let *(i)* hold.

First, let d'' be the maximal element of $L''_n(f,g)$. Then we introduce d using the substitution (a_2). Therefore $d \in \mathbb{D}_n$ but $d \notin L''_n(f,g)$, since $d > d''$. The assumption $d \in \mathbb{D}''_n$ yields:

$$f(d)g\left(\frac{n}{d}\right) < f(d'')g\left(\frac{n}{d''}\right) \tag{22}$$

On the other hand, because of *(i)* and (a_2), (a_{21}) holds and therefore (21) holds, which contradicts to the above inequality. Therefore the assumption $d \in \mathbb{D}''_n$ is wrong. Hence $d \in \mathbb{D}'_n$. But if $d \in \mathbb{D}'_n$ then (21) yields $(*_1)$.

Second, let d'' be the minimal element of $l''_n(f,g)$. Then we introduce d using the substitution (a_1). Therefore $d \in \mathbb{D}_n$ but $d \notin l''_n(f,g)$, since $d < d''$. The assumption $d \in \mathbb{D}''_n$ yields:

$$f(d)g\left(\frac{n}{d}\right) > f(d'')g\left(\frac{n}{d''}\right) \tag{23}$$

On the other hand, because of *(i)* and (a_1), (a_{11}) holds and therefore (20) holds, which contradicts to the above inequality. Therefore the assumption $d \in \mathbb{D}''_n$ is wrong. Hence $d \in \mathbb{D}'_n$. But if $d \in \mathbb{D}'_n$ then (20) yields $(*_2)$.

Let *(ii)* hold.

First, let d'' be the minimal element of $L''_n(f,g)$. Then we introduce d using the substitution (a_1). Therefore $d \in \mathbb{D}_n$ but $d \notin L''_n(f,g)$, since $d < d''$. The assumption $d \in \mathbb{D}''_n$ yields (22). On the other hand, because of *(ii)* and (a_1), (a_{12}) holds and therefore (21) holds, which contradicts to (22). Therefore the assumption $d \in \mathbb{D}''_n$ is wrong. Hence $d \in \mathbb{D}'_n$. But if $d \in \mathbb{D}'_n$ then (21) yields $(*_1)$.

Second, let d'' be the maximal element of $l''_n(f,g)$. Then we introduce d using the substitution (a_2). Therefore $d \in \mathbb{D}_n$ but $d \notin l''_n(f,g)$, since $d > d''$. The assumption $d \in \mathbb{D}''_n$ yields (23). On the other hand, because of *(ii)* and (a_2), (a_{22}) holds and therefore (20) holds, which contradicts to (23). Therefore the assumption $d \in \mathbb{D}''_n$ is wrong. Hence $d \in \mathbb{D}'_n$. But if $d \in \mathbb{D}'_n$ then (20) yields $(*_2)$.

Theorem 1 is proved. □

Corollary 1. *Under the conditions of Theorem 1 the relations:*

$$L'_n(f,g) \subseteq L_n(f,g)$$

$$l'_n(f,g) \subseteq l_n(f,g)$$

hold.

Remark 1. *If the inequalites $(*_1)$ and $(*_2)$ are strict, then Theorem 1 implies:*

$$L'_n(f,g) = L_n(f,g) \tag{24}$$

$$l'_n(f,g) = l_n(f,g) \tag{25}$$

but when $(*_1)$ and $(*_2)$ are equalities we have the relations:

$$L_n(f,g) = L'_n(f,g) \cup L''_n(f,g)$$

$$l_n(f,g) = L'_n(f,g) \cup l''_n(f,g)$$

Remark 2. *If $\mathbb{D}''_n = \emptyset$ then (24) and (25) stay valid.*

The following assertion gives a simple connection between L'_n and l'_n.

Lemma 1. *Let $n \in \mathbb{N}$ and $f, g \in \mathbb{M}$. Then it is valid:*

(j) If $d \in L'_n(f,g)$ then $\frac{n}{d} \in l'_n(f,g)$

(jj) If $d \in l'_n(f,g)$ then $\frac{n}{d} \in L'_n(f,g)$

Proof. From the definition of $\mathbb{D}'_n$, if $d \in \mathbb{D}'_n$ then $\frac{n}{d} \in \mathbb{D}'_n$ and $\gcd(d, \frac{n}{d}) = 1$. When $f, g \in \mathbb{M}$ the equalities:

$$f\left(\frac{n}{d}\right) = \frac{f(n)}{f(d)};\ g\left(d\right) = \frac{g(n)}{g\left(\frac{n}{d}\right)}$$

hold, since $\gcd(d, \frac{n}{d}) = 1$. Hence:

$$f\left(\frac{n}{d}\right) g(d) = \frac{f(n)g(n)}{f(d)g\left(\frac{n}{d}\right)}$$

Therefore *(j)* and *(jj)* follow immediately from the above equality, since when d runs over $\mathbb{D}'_n$, then $\frac{n}{d}$ runs over $\mathbb{D}'_n$, too.

Lemma 1 is proved. $\square$

Our second main result is:

Theorem 2. *Under the conditions of Theorem 1 it is valid:*

(k) There exists at least one element $d \in L_n(f,g)$ such that $\frac{n}{d} \in l_n(f,g)$

(kk) There exists at least one element $d \in l_n(f,g)$ such that $\frac{n}{d} \in L_n(f,g)$

Remark 3. *According to Theorem 1, for (k) such elements are contained at least in $L'_n(f,g)$, while for (kk) such elements are contained at least in $l'_n(f,g)$.*

Below we must give the following important

Observation. *Property $\mathbb{S}$ is essential for the validity of Theorem 1 and Theorem 2. Indeed, let φ be Euler's totient function and τ be the function that for a given $n \in \mathbb{N}$ coincide with the number of all the divisors of n (i.e. $\tau(n) = \sum\limits_{d|n} 1$). Then $\varphi, \tau \in \mathbb{M}$ but the ordered pair (φ, τ) does not have the property $\mathbb{S}$, since*

$$H^{\varphi,\tau}_{2,m}(k) = \varphi(2^k)\tau(2^{m-k})$$

is not a monotone function with respect to k . Let $n = 32$. Then all divisors of n are: $1, 2, 4, 8, 16, 32$ and $\varphi(1)\tau(32) = 6, \varphi(2)\tau(16) = 5, \varphi(4)\tau(8) = 8, \varphi(8)\tau(4) = 12, \varphi(16)\tau(2) = 16, \varphi(32)\tau(1) = 16$. Therefore, for $n = 32$

$$\min_{d|n} \varphi(d)\tau\left(\frac{n}{d}\right) = \min_{d\in\mathbb{D}_{32}} \varphi(d)\tau\left(\frac{n}{d}\right) = \varphi(2)\tau\left(\frac{32}{2}\right) = 5.$$

Hence, for $n = 32$ we have the situation when $\min\limits_{d\in\mathbb{D}_n} \varphi(d)\tau\left(\frac{n}{d}\right)$ is obtained for $n = 2 \in \mathbb{D}''_{32}$ and it is not obtained for any $d \in \mathbb{D}'_{32}$. Therefore, for $n = 32$, Theorem 1 is not valid for the ordered pair (φ, τ), but moreover, Theorem 2 is also not valid, since for $n = 32$ and $d = 32$ it is easy to see that $d \in L_{32}(\varphi, \tau)$ & $d \in \mathbb{D}'_{32}$, but $1 = \frac{n}{d} \notin l_{32}(\varphi, \tau)$.

Let $\sigma(n) = \sum\limits_{d|n} d$ (i.e. $\sigma(n)$ is the sum of all divisors of n, including 1 and n). Then the ordered pair (φ, σ) has the property $\mathbb{S}$, since $\sigma \in \mathbb{M}$ and

$$H^{\varphi,\sigma}_{p,m}(k) = \varphi(2^k)\sigma(2^{m-k})$$

is a decreasing function for all $p \in \mathbb{P}$ and $m \in \mathbb{Z}^+$ with respect to $k \in [0, m] \cap \mathbb{Z}^+$. Therefore for the ordered pair (φ, σ) Theorem 1 and Theorem 2 hold. The same observation is valid for the ordered pair (φ, ψ), where $\psi \in \mathbb{M}$ is Dedekind's function, for $t \geq 2$ given by:

$$\psi(t) = t \prod_{i=1}^{\gamma} \left(1 + \frac{1}{p_i}\right)$$

where $p_i, i = 1, \ldots, \gamma$, are all prime divisors of t and $\psi(1) = 1$.

Remark 4. *We note that the ordered pairs* (σ, φ) *and* (ψ, φ) *have the property* $\mathbb{S}$ *too, but this time their corresponding functions:*

$$H^{\sigma,\varphi}_{p,m}(k) = \sigma(2^k)\varphi(2^{m-k})$$

and

$$H^{\psi,\varphi}_{p,m}(k) = \psi(2^k)\varphi(2^{m-k})$$

are increasing with respect to $k \in [0, m] \cap \mathbb{Z}^+$*. Moreover, if for* $f, g \in \mathbb{M}$ *the ordered pair (g,f) satisfies (i) , then the ordered pair (f,g) satisfies (ii) and vice versa.*

One may verify that the ordered pair (σ, τ) has the property $\mathbb{S}$ too.

Let $x \in \mathbb{N}, x > 1$, admit the cannonical factorization in primes of the form:

$$x = \prod_{i=1}^{s} p_i^{m_i} \tag{26}$$

where $s \in \mathbb{N}$, $m_i \in \mathbb{Z}^+, i = 1, \dots, s$ and $\sum_{i=1}^{s} m_i > 0$. Then every divisor d of x has the representation:

$$d = \prod_{i=1}^{s} p_i^{k_i} \tag{27}$$

where $0 \le k_i \le m_i,\ i = 1, \dots, s$. For such d, when $f, g \in \mathbb{M}$, from (26) and (27) we obtain:

$$f(d)g\left(\frac{x}{d}\right) = \prod_{i=1}^{s} f(p_i^{k_i})g(p_i^{m_i-k_i}) = \prod_{i=1}^{s} H^{f,g}_{p_i,m_i}(k_i) \tag{28}$$

Let the ordered pair (f, g) have the property $\mathbb{S}$, satisfying *(ii)*. Then each one of the factors $H^{f,g}_{p_i,m_i}(k_i)$ takes its maximal value when $k_i = 0,\ i = 1, \dots, s$. Hence:

$$\max_{d \in \mathbb{D}_x} f(d)g\left(\frac{x}{d}\right) = \prod_{i=1}^{s} g(p_i^{m_i}) = g(x) \tag{29}$$

because of (26) and (28).

On the other hand, since x and 1 belong to $\mathbb{D}'_x$, we obtain from Theorem 1 and Lemma 1, *(j)*, that:

$$\min_{d \in \mathbb{D}_x} f(d)g\left(\frac{x}{d}\right) = f(x) \tag{30}$$

Therefore, (29) and (30) yield:

$$f(x) \le g(x). \tag{31}$$

The last inequality is valid for $x = 1$ too, since $f, g \in \mathbb{M}$ implies $f(1) = g(1) = 1$.

If the ordered pair (f, g) has the property $\mathbb{S}$, satisfying *(i)*, then because of Remark 4 the ordered pair (g, f) satisfies *(ii)* and applying the same reasoning as above, we conclude that:

$$g(x) \le f(x) \tag{32}$$

for every $x \in \mathbb{N}$.

Thus we proved the third main result of the paper:

Theorem 3. *Let $f, g \in \mathbb{M}$. The necessary condition the ordered pair (f, g) to have the property $\mathbb{S}$ is: for every $x \in \mathbb{N}$*

to be fulfilled (32) for the case (i)

and

to be fulfilled (31) for the case (ii).

Remark 5. *We note that the above necessary condition is not a sufficient one.*

Now, it is clear that when *(i)* holds, then it is fulfilled:

$$\max_{d \in \mathbb{D}_x} f(d) g\left(\frac{x}{d}\right) = f(x) \tag{33}$$

$$\min_{d \in \mathbb{D}_x} f(d) g\left(\frac{x}{d}\right) = g(x) \tag{34}$$

Thus we proved the fourth main result of the paper:

Theorem 4. *Let $x \in \mathbb{N}$, $f, g \in \mathbb{M}$ and let the ordered pair (f, g) have the property $\mathbb{S}$. If (i) holds then (33) and (34) are fulfilled, but if (ii) holds then the relations*

$$\max_{d \in \mathbb{D}_x} f(d) g\left(\frac{x}{d}\right) = g(x)$$

and

$$\min_{d \in \mathbb{D}_x} f(d) g\left(\frac{x}{d}\right) = f(x)$$

are fulfilled.

Corollary 2. *Let $x \in \mathbb{N}$, $f, g \in \mathbb{M}$ and let the ordered pair (f, g) have the property $\mathbb{S}$. If (i) holds then $x \in L_x(f, g)$ and $1 \in l_x(f, g)$, but if (ii) holds then $1 \in L_x(f, g)$ and $x \in l_x(f, g)$.*

Remark 6. *Since $\gcd(1, x) = 1$, the elements 1 and x belong to $\mathbb{D}'_x$.*

Let $x \in \mathbb{N}$ be given by (26). Up to now our considerations were related to the set D_x. Below, we will consider the set D^*_x, supposing that x is such that $D^*_x \neq \emptyset$ (i.e. $x > 1$ & $x \notin \mathbb{P}$). Let us note that if d runs over D^*_x then $\frac{x}{d}$ runs over D^*_x too. Each $d \in D^*_x$ admits the representation(27) with

$$\sum_{i=1}^{s} k_i > 0 \tag{35}$$

and (28) holds.

Let $f, g \in \mathbb{M}$ and the ordered pair (f, g) satisfies *(ii)*. Then each one of $H^{f,g}_{p_i,m_i}(k)$ is a decreasing function with respect to $k \in [0, m_i] \cap \mathbb{Z}^+$, $i = 1, \ldots, s$ and (35) holds. Therefore to obtain

$$\max_{d \in \mathbb{D}^*_x} f(d) g\left(\frac{x}{d}\right)$$

we must consider the case when exactly for one of p_i, $i = 1, \ldots, s$ (for example for p_j), we have $k_j = 1$ and for all others we have $k_i = 0$. In this case we have (see (28)):

$$\prod_{i=1}^{s} H^{f,g}_{p_i,s_i}(k_i) = f(p_j) g(p_j^{m_j - 1}) \prod_{\substack{i=1 \\ i \neq j}}^{s} g(p_i^{m_i}) = f(p_j) g\left(\frac{x}{p_j}\right)$$

It remains only to take the maximal of the numbers: $f(p_j)g\left(\frac{x}{p_j}\right)$, $j = 1, \ldots, s$, to obtain:

$$\max_{d\in\mathbb{D}_x^*} f(d)g\left(\frac{x}{d}\right) = \max_{1\le j\le s} f(p_j)g\left(\frac{x}{p_j}\right)$$

If we want to obtain

$$\min_{d\in\mathbb{D}_x^*} f(d)g\left(\frac{x}{d}\right)$$

then because of the fact that $H^{f,g}_{p_i,m_i}(k)$ are decreasing functions with respect to $k \in [0, m_i] \cap \mathbb{Z}^+$, $i = 1, \ldots, s$, we must take exactly for one of p_i, $i = 1, \ldots, s$ (for example for p_j), $k_j = m_j - 1$ and for the others $k_i = m_i$. Hence:

$$\min_{d\in\mathbb{D}_x^*} f(d)g\left(\frac{x}{d}\right) = \min_{1\le j\le s} f\left(\frac{x}{p_j}\right)g(p_j)$$

Let $f, g \in \mathbb{M}$ and the ordered pair (f, g) satisfies *(i)*. Then the ordered pair (g, f) satisfies *(ii)* (see Remark 4). Hence (as above):

$$\max_{d\in\mathbb{D}_x^*} f(d)g\left(\frac{x}{d}\right) = \max_{1\le j\le s} f\left(\frac{x}{p_j}\right)g(p_j);$$

$$\min_{d\in\mathbb{D}_x^*} f(d)g\left(\frac{x}{d}\right) = \min_{1\le j\le s} f(p_j)g\left(\frac{x}{p_j}\right)$$

Thus we proved the fifth main result of the paper:

Theorem 5. *Let $x \in \mathbb{N}\setminus\mathbb{P}, x > 1, f, g \in \mathbb{M}$ and the ordered pair (f, g) has the property $\mathbb{S}$. If (i) holds then:*

$$\max_{d\in\mathbb{D}_x^*} f(d)g\left(\frac{x}{d}\right) = \max_{p} f\left(\frac{x}{p}\right)g(p); \min_{d\in\mathbb{D}_x^*} f(d)g\left(\frac{x}{d}\right) = \min_{p} f(p)g\left(\frac{x}{p}\right),$$

but if (ii) holds then:

$$\max_{d\in\mathbb{D}_x^*} f(d)g\left(\frac{x}{d}\right) = \max_{p} f(p)g\left(\frac{x}{p}\right); \min_{d\in\mathbb{D}_x^*} f(d)g\left(\frac{x}{d}\right) = \min_{p} f\left(\frac{x}{p}\right)g(p)$$

where p runs over all prime divisors of x.

Finally, we shall consider the particular case $g \equiv 1$. In this case, property $\mathbb{S}$ for the ordered pair $(f, 1)$ is expressed by the condition:

$\tilde{\mathbb{S}}:\ \forall p \in \mathbb{P}\ \tilde{H}^f_p(k) \stackrel{\text{def}}{=} f(p^k)$ is increasing (decreasing) function with respect to $k \in \mathbb{Z}^+$.

As a result we obtain the following theorem which is a direct corollary from Theorem 1

Theorem 6. *Let $f \in \mathbb{M}$ satisfy the condition $\tilde{\mathbb{S}}$. If for a given $n \in \mathbb{N}$, $\mathbb{D}''_n \neq \emptyset$ then the inequalities:*

$$\max_{d\in\mathbb{D}''_n} f(d) \le \max_{d\in\mathbb{D}'_n} f(d); \min_{d\in\mathbb{D}'_n} f(d) \le \min_{d\in\mathbb{D}'_n} f(d)$$

hold.

Let us denote:

$$L_n(f) \stackrel{\text{def}}{=} L_n(f,1);\ l_n(f) \stackrel{\text{def}}{=} l_n(f,1);\ L'_n(f) \stackrel{\text{def}}{=} L'_n(f,1);\ l'_n(f) \stackrel{\text{def}}{=} l'_n(f,1)$$

Then the following analogue of Corollary 1 is valid.

Corollary 3. *Under the conditions of Theorem 6 the relations:*

$$L'_n(f) \subseteq L_n(f); l'_n(f) \subseteq l_n(f)$$

hold.

Analogue of Lemma 1 now is:

Lemma 2. *Let $n \in \mathbb{N}$ and $f \in \mathbb{M}$. Then it is valid:*

($\tilde{j}$) If $d \in L'_n(f)$ then $\frac{n}{d} \in l'_n(f)$

($\tilde{j}\tilde{j}$) If $d \in l'_n(f)$ then $\frac{n}{d} \in L'_n(f)$

Finally, analogue to Theorem 2 is the following:

Theorem 7. *Under the conditions of Theorem 6 it is valid:*

($\tilde{k}$) There exists at least one element $d \in L_n(f)$ such that $\frac{n}{d} \in l_n(f)$

($\tilde{k}\tilde{k}$) There exists at least one element $d \in l_n(f)$ such that $\frac{n}{d} \in L_n(f)$.

Remark 7. *Let $f \in \mathbb{M}$ satisfy the condition $\tilde{\mathbb{S}}$. Due to $f(1) = 1$, if*
(v) $\tilde{H}_p^f(k)$ is an increasing function with respect to $k \in \mathbb{Z}^+$
holds, then $\forall m \in \mathbb{N}$ $f(m) \geq 1$,
but if
(vv) $\tilde{H}_p^f(k)$ is a decreasing function with respect to $k \in \mathbb{Z}^+$
holds, then $\forall m \in \mathbb{N}$ $f(m) \leq 1$.

Remark 7 is due to Theorem 3 with $g \equiv 1$.
If *(v)* holds then for $d \in \mathbb{D}'_n$ we obtain

$$f(n) = f\left(d.\frac{n}{d}\right) = f\left(d\right) f\left(\frac{n}{d}\right) \geq f(d)$$

Hence:

$$n \in L'_n(f) \subseteq L_n(f)$$

while

$$1 \in l'_n(f) \subseteq l_n(f)$$

If *(vv)* holds then for $d \in \mathbb{D}'_n$ we obtain

$$f(n) = f\left(d.\frac{n}{d}\right) = f\left(d\right) f\left(\frac{n}{d}\right) \leq f(d)$$

Hence:

$$n \in l'_n(f) \subseteq l_n(f)$$

while

$$1 \in L'_n(f) \subseteq L_n(f)$$

The above is due to Theorem 4 with $g \equiv 1$ and Corollary 2.
The analogue of Theorem 5, which is due to $g \equiv 1$, is

Theorem 8. *Let $x \in \mathbb{N} \setminus \mathbb{P}, x > 1, f \in \mathbb{M}$ and f satisfies the condition $\tilde{\mathbb{S}}$. If (v) holds then:*

$$\max_{d \in \mathbb{D}_x^*} f(d) = \max_p f\left(\frac{x}{p}\right); \min_{d \in \mathbb{D}_x^*} f(d) = \min_p f(p),$$

but if (vv) holds then:

$$\max_{d \in \mathbb{D}_x^*} f(d) = \max_p f(p); \min_{d \in \mathbb{D}_x^*} f(d) = \min_p f\left(\frac{x}{p}\right)$$

where p runs over all prime divisors of x.

Remark 8. *We note that each of the well-known functions φ, ψ, σ, τ, which belong to $\mathbb{M}$, satisfies the condition $\tilde{\mathbb{S}}$ (case (v)). Therefore for these functions Theorem 6, Theorem 7 and Corollary 3 are valid.*

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