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FERMATIAN ANALOGUES OF GOULD'S GENERALIZED BERNOULLI POLYNOMIALS

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Abstract: Some results of Gould for the ordinary Bernoulli and Euler polynomials are extended to analogous results built upon the Fermatian exponentials.

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1. Introduction

It is of interest to consider generalized Bernoulli and Euler polynomials analogous to those of Gould [3]. Let

$$\frac{tC(tx)}{C(t)-1} = \sum_{k=0}^{\infty} B_{kz}(x,c)t^k / \underline{z}_k! \tag{1.1}$$

and

$$\frac{2C(tx)}{C(t)+1} = \sum_{k=0}^{\infty} E_{kz}(x,c)t^k / \underline{z}_k!$$

define $B_{kz}(x,c)$ and $E_{kz}(x,c)$, in which

$$C(t) = E_z(ct) \tag{1.2}$$

where $E_z(t)$ is the Fermatian exponential.

This is analogous to the ordinary situation where

$$c^t = e^{ct} \text{ if } C = e^C .$$

In Gould's work, $C = b/a$, in which

$$a = \frac{1}{2}\left(1+\sqrt{5}\right) \text{ and } b = \frac{1}{2}\left(1-\sqrt{5}\right)$$

are the roots of $x^2 - x - 1 = 0$.

2. Fermatian Numbers

We can define [7] the *n*-th reduced Fermatian number in terms of

$$\underline{q}_n = \begin{cases} -q^n \underline{q}_{-n} & (n < 0) \\ 1 & (n = 0) \\ 1 + q + q^2 + \ldots + q^{n-1} & (n > 0) \end{cases} \tag{2.1}$$

so that

$$\underline{1}_n = n,$$

and

$$\underline{1}_n ! = n!,$$

where

$$\underline{q}_n ! = \underline{q}_n \underline{q}_{n-1} \cdots \underline{q}_1 . \tag{2.2}$$

Accordingly, we define

$$E_z(x) = \sum_{n=0}^{\infty} x^n / \underline{z}_n ! \tag{2.3}$$

Note that

$$E_1(x) = e^x .$$

3. Results of Gould and Hoggatt

Incidentally, Gould's C and Hoggatt's C_{nk} [6] can be related when $p = -q = 1$ in the equality $x^2 - px + q = 0$:

$$C = b \lim_{k\to\infty} C_{k+1,k+1} / C_{kk} \tag{3.1}$$

Proof:

$$C_{kk} = \frac{F_{k-1}F_{k-2}\ldots F_1}{F_1 F_2 \ldots F_{k-1} F_k} \quad \text{when } p = -q = 1$$
$$= 1/F_k$$

where F_k is the k^{th} Fibonacci number. Then

$$b \lim_{k\to\infty} C_{k+1,k+1} / C_{kk} = b \lim_{k\to\infty} F_k / F_{k+1}$$
$$= b/a, \qquad \text{from Vorob'ev [4].}$$

4. Main Result

From (1.1) we get

$$\sum_{k=0}^{\infty} B_{kz}(x.c)t^k / \underline{z}_k! \quad = \frac{tE_z(ctx)}{E_z(ct)-1}$$

$$= \frac{1}{c}\frac{ctE_z(ctx)}{E_z(ct)-1}$$

$$= \frac{1}{c}\sum_{k=0}^{\infty} B_{kz}(x)(ct)^k / \underline{z}_k!$$

which gives

$$B_{kz}(x,c) \quad = B_{kz}(x)c^{k-1} \tag{4.1}$$

as a relation between the analogues of the generalized and ordinary Fermatian Bernoulli polynomials. A similar relation can be found for Euler polynomials which the interested reader might like to pursue.

When $z = 1$ we get the corresponding relation for ordinary Bernoulli polynomials

$$B_k(x,c) \quad = B_{kz}(x)(\log c)^{k-1}, \tag{5.1}$$

which agrees with Gould.

5. Concluding Comments

The paper by Gould [3] has a number of interesting and elegant relationships among Bernoulli, Euler numbers with Fibonacci, Lucas numbers. Carlitz too explored some of these types of analogues [1, 2]. Another avenue of research would be to find Fermatian analogues of binomial functions and the bracket function [4, 5].

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