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INTEGER STRUCTURE ANALYSIS OF ODD POWERED TRIPLES: THE SIGNIFICANCE OF TRIANGULAR VERSUS PENTAGONAL NUMBERS

J.V. Leyendekkers
The University of Sydney, 2006 Australia

A.G. Shannon
Warrane College, the University of New South Wales,
Kensington, NSW 1465, Australia

Abstract: Structural constraints prevent the difference of two odd cubes ever equaling an even cube. This is illustrated from the row structure of the modular ring Z_4. The critical structure factor is that the rows of integers, N^2, with $3 \mid N^2$, follow the triangular numbers, whereas $3 \nmid N^2$ rows follow the pentagonal numbers. This structural characteristic is the reason for the importance of primitive Pythagorean triples (in which either the smallest odd component or the even component always has a factor 3).

Keywords: Primitive Pythagorean triples, triangular numbers, pentagonal numbers, modular rings, cubic triples.

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1. Introduction

The key to the analysis of primitive Pythagorean triples (pPts) is that in the modular ring Z_4 (Table 1) the rows of the squares of the two odd components are compatible with the row of the square of the even component [6].

The factor 3 is important. The row of an odd integer, N^2, characterised by $3 \mid N^2$, follows the triangular numbers [8,11], whereas $3 \nmid N^2$ follows the pentagonal numbers [3,9]. This intriguing integer-structure characteristic seems to be critical when the powers of the triples exceed the square. We note, in passing, the caution of Harkleroad: “If you look at some numbers, you may very well find Fibonacci numbers, or square numbers, or prime numbers, or any other favorite kind of number you have. But unless you can also find a particular reason why numbers of a certain type *should* appear, you’re quite likely just turning up a coincidence” [4].

Since the structure of higher odd powers has the same form as the cube, cubes can be considered as representative of odd powers in general [5, 7]. The aim of this paper then is to provide examples of how the incompatible factor-structure of cubes prevents the sum of two cubes from equalling a cube.

Row	$f(r)$	$4r_0$	$4r_1+1$	$4r_2+2$	$4r_3+3$
	Class	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$
0		0	1	2	3
1		4	5	6	7
2		8	9	10	11
3		12	13	14	15
4		16	17	18	19
5		20	21	22	23
6		24	25	26	27
7		28	29	30	31

Table 1: Rows of Z_4

2. The Incompatibility of the Rows of Cubic Triples

Odd cubes belong to Classes $\bar{1}_4$ or $\bar{3}_4$, but even cubes belong only to $\bar{0}_4$ (Table 1). Hence, if we assume that for the integers *a, b, c*, with *a* and *c* odd, and *b* even

$$c^3 = a^3 + b^3 \tag{2.1}$$

then $a, c \in \bar{1}_4$:

$$4R_1 + 1 = 4R_1' + 1 + 4R_0 \tag{2.2}$$

and thus

$$R_1 - R_1' = R_0 . \tag{2.3}$$

Observe that *a, c* must fall in the same class because

$$4R_1 + 1 - 4R_3 - 3 = 4(R_1 - R_3 - 1) + 2 \tag{2.4}$$

or

$$4R_3 + 3 - 4R_1 - 1 = 4(R_3 - R_1) + 2 \tag{2.5}$$

and Class $\bar{2}_4$ has no powers.

When $a, c \in \bar{3}_4$,

$$4R_3 + 3 = 4R_3' + 3 + 4R_0 \tag{2.6}$$

so that

$$R_3 - R_3' = R_0 . \tag{2.7}$$

The rows of cubes are functions of the pentagonal numbers, or, when 3 is a factor of the cube, the triangular numbers (Table 2).

When the cube is even the rows have the forms $2^q(4R_1+1), 2^q(4R_3+3)$ or 2^q, q = 1, 2 , 3, … R_1 and R_3 are the rows of the odd factors cubed and hence conform to the functions in Table 2. For example, $6^3 = 2^3 \times 27$. Thus, $R_0 = 2(27)$ (q = 1) with $27 = 4R_3 + 3$, $R_3 = 6$ ($n = 0$; $f(n)$ no.6 in Table 2).

(i) $3 \mid b \;\&\; 3 \nmid c, a$ for b^3

In this case, R_1 is given by $f(n)$ no.5 and R_3 by $f(n)$ no.6 in Table 2, so that

No.	Class of N	Row of N	n	Row of cube	Class of n	R_1 parity
1	$\bar{1}_4$ $3 \nmid N$	$\frac{3}{2}n$	even	$3n(3n+1)(6n+1)+\frac{1}{2}(3n)$	$\bar{2}_4$ $\bar{0}_4$	odd even
2	$\bar{1}_4$ $3 \nmid N$	$\frac{1}{2}(3n-1)$	odd	$3n(3n-1))(6n-1)+\frac{1}{2}(3n-1)$	$\bar{1}_4$ $\bar{3}_4$	odd even
3	$\bar{3}_4$ $3 \nmid N$	$\frac{1}{2}(3n-2)$	even	$3n(3n-1))(6n-1)+\frac{1}{2}(3n-2)$	$\bar{2}_4$ $\bar{0}_4$	even odd
4	$\bar{3}_4$ $3 \nmid N$	$\frac{1}{2}(3n-1)$	odd	$3n(3n+1))(6n+1)+\frac{1}{2}(3n-1)$	$\bar{1}_4$ $\bar{3}_4$	odd even
5	$\bar{1}_4$ $3 \mid N$	$\frac{1}{2}(3n+1)$	odd	$3(2+9n(n+1))(2n+1)\,)+\frac{1}{2}(3n+1)$	$\bar{1}_4$ $\bar{3}_4$	even odd
6	$\bar{3}_4$ $3 \mid N$	$\frac{3}{2}n$	even	$3(2+9n(n+1))(2n+1)+\frac{1}{2}(3n)$	$\bar{2}_4$ $\bar{0}_4$	odd even

Table 2: Rows of odd cubes (e: even; o: odd)

$$\begin{aligned} R_0 &= 2^q(4R_1+1) \\ &= 27 \times 2^q(f(m)) \end{aligned}$$

with

$$f(m) = 4\left(2m^3+3m^2+\tfrac{1}{2}(3m-1)\right)+3 \in \bar{3}_4,\ m \text{ odd}, \tag{2.8}$$

or

$$\begin{aligned} R_0 &= 2^q(4R_3+3) \\ &= 27 \times 2^q(f(m)) \end{aligned}$$

with

$$f(m) = 4\left(2m^3+3m^2+\tfrac{1}{2}(3m)\right)+1 \in \bar{1}_4,\ m \text{ even}. \tag{2.9}$$

Some examples occur in Table 3.

b	6	12	18	24	30	36	42	48	54	60	66	72	78
m	0	0	1	0	2	1	3	0	4	2	5	1	6
q	2	4	1	7	1	4	1	8	1	4	1	7	3

Table 3

R_1 and R_1' will equal the $f(n)$ numbers 1 to 4 in Table 2 according to the Class of c and a, which, as noted above, must fall in the same class. However, a further restriction is that c, a must both fall in $f(n)$ numbers 1 or 2 when in Class $\bar{1}_4$ or in $f(n)$ numbers 3 or 4 when in Class $\bar{3}_4$. For instance, suppose c falls in $f(n)$ (no.1) and a falls in $f(n)$ (no.2), then

$$R_1 - R_1' = 9f(n,n') + 2. \tag{2.10}$$

so that $3 \nmid R_1 - R_1'$ and R_0 is incompatible. Possible compatible functions for $R_1 - R_1'$ are listed in Table 4.

Class of N	**f(n), $f(n')$** (Table 2)	$R_1 - R_1'$	**n, n' parity**
$\bar{1}_4$	1	$\frac{9}{2}(n-n')\left\{4\left(3\left(n^2+n'^2+nn'\right)+\frac{3}{2}\left(n+n'\right)\right)+1\right\}$	even
$\bar{1}_4$	2	$\frac{9}{2}(n-n')\left\{4\left(3\left(n^2+n'^2+nn'\right)-\frac{3}{2}\left(n+n'\right)\right)+1\right\}$	odd
$\bar{3}_4$	3	$\frac{9}{2}(n-n')\left\{4\left(3\left(n^2+n'^2+nn'\right)-\frac{3}{2}\left(n+n'\right)\right)+1\right\}$	even
$\bar{3}_4$	4	$\frac{9}{2}(n-n')\left\{4\left(3\left(n^2+n'^2+nn'\right)+\frac{3}{2}\left(n+n'\right)\right)+1\right\}$	odd

Table 4

Since $(n-n')$ could have a factor $(2^q \times 3)$ there could be a match with R_0 provided the $f(m)$ of Equations (2.8) and (2.9) are compatible with $R_1 - R_1'$. When R_0 has the odd factor in Class $\bar{1}_4$, $f(m)$ belongs to Class $\bar{3}_4$, where as all the ($R_1 - R_1'$) functions are in Class $\bar{1}_4$ (Table 4). Thus we need only consider R_0 with the odd factor in Class $\bar{3}_4$.

This yields Equation (2.9) for R_0 since $R_1 - R_1' = \frac{9}{2}(n-n')f(n,n')$ (Table 4). $f(n,n') \in \bar{1}_4$ and the row of $f(n,n')$ has a factor of 3. Thus, for compatibility, the m of Equation (2.9) must have $3 \mid m$.

In this case, $f(m)$ would have a factor of at least 3^2, whereas if n, n' are replaced by the appropriate class function from Table 2, then there is always a mismatch with the factor 3 so that R_0 and ($R_1 - R_1'$) are never compatible. Even so, the $2^q \times 27$ of R_0 must simultaneously match the $\frac{9}{2}(n-n')$ of ($R_1 - R_1'$).

(ii) $3 \mid a \,\&\, 3 \nmid c, b$.

In this case the $f(n)$ for the row of a will be given by numbers 5 or 6 of Table 2. If we consider Class $\bar{1}_4$ with R_1 given by the $f(n)$ of number 1 and R_1' given by the $f(n)$ of number 5 in Table 2, then

$$R_1 = 3n(3n+1)(6n+1) + \tfrac{1}{2}(3n), \quad n \text{ even}, \tag{2.11}$$

and the row of a is given by

$$R_1' = 3\left(2 + 9n'(n'+1)\right)(2n'+1) + \tfrac{1}{2}(3n'+1), \quad n \text{ odd}, \tag{2.12}$$

so

$$R_1 - R_1' = 27\left(2n^3 - 2n'^3 + n^2 - 3n'^2\right) + \tfrac{1}{2}\left(9(n - 9n') - 13\right), \tag{2.13}$$

n even, n' odd.

We take the odd factor for $b \in \bar{1}_4$ so that if we use $f(n)$, number 1, in Table 2, then

$$R_0 = 2^q(4f(m)+1), \quad m \text{ even} \tag{2.14}$$

with

$$f(m) = 9(6m^3 + 3m^2 + \tfrac{1}{2}m).$$

Thus

$$\frac{R_0}{2^q} \in \bar{1}_4 .$$

Since R_1 and R_1' must have the same parity because R_0 is even (from Table 2), then

$$n, n' \in \left\{(\bar{2}_4, \bar{3}_4), (\bar{0}_4, \bar{1}_4)\right\}$$

If we substitute $n = 4r_0, n' = 4r_1 + 1$ into Equation (2.13), then we obtain

$$R_1 - R_1' = 2(4f(r_1, r_0) + 1) \tag{2.15}$$

where

$$f(r_1, r_0) = 27\left(16(r_0^3 - r_1^3) - 18r_1^2 + 2r_0^2 - \frac{27}{2}\left(r_1 + \frac{r_0}{3}\right)\right) - 23 . \tag{2.16}$$

Even though we can take $q = 1$, and $r_1 r_0$ even with $3 \mid r_0$, the row of is not divisible by 3, whereas the row of $\frac{1}{2}R_0$ can be.

If we let $n = 4r_2 + 2$ and $n' = 4r_3 + 3$, then

$$R_1 - R_1' = 2(4f(r_2, r_3) + 1) \tag{2.17}$$

where

$$f(r_2, r_3) = 9\left(48(r_2^3 - r_3^3) + 71r_2^2 - 126r_3^2 + \frac{1}{4}(169r_2 - 441r_3)\right) - 221.$$

Thus, even though $\frac{1}{2(R_1 - R_1')} \in \overline{1}_4$, the row has no factor of 3, and so $(R_1 - R_1')$ is incompatible with R_0.

If we take the odd factor of $R_0 \in \overline{3}_4$, then by using *f*(*n*) number 3 in Table 2, we have

$$R_0 = 2^q\left(4\left(54m^3 - 27m^2 + \frac{1}{2}(9m) - 1\right) + 3\right) \tag{2.18}$$

so that $\frac{R_0}{2^q} \in \overline{3}_4$ which is incompatible with $(R_1 - R_1')$.

3. No Component has a Factor of 3

If R_0 has the odd factor in Class $\overline{1}_4$,
then

$$R_0 = 2^q(4R_1 + 1) \tag{3.1}$$

with R_1 connected with *f*(*n*) number 2 in Table 2,

$$R_0 = 2^q\left(4\left(54m^3 - 27m^2 + \frac{1}{2}(9m - 1)\right) + 1\right), \ m \text{ odd.} \tag{3.2}$$

The row of $\frac{R_0}{2^q}$ is even when $m \in \overline{1}_4$ and odd when $m \in \overline{3}_4$, but does not have 3 as a factor, whereas the rows of $\frac{1}{2}(R_1 - R_1')$ have 3 as a factor (Table 4). Again various combinations never lead to compatibility between $(R_1 - R_1')$ and R_0.

These results can be compared with those for the square triple $(c^2 - a^2 = b^2)$. As shown previously [6], for a pPt $(c^2 - a^2 = b^2)$ either *a* or *b* always has 3 as a factor. Furthermore, all odd squares belong to $\overline{1}_4(4r_1 + 1)$, unlike odd powers N^m which fall in the same class as *N*.

However, it is the unique row structure of integers with a factor of 3 that is the critical point. That is, rows of N^2 with $3|N$ follow the triangular numbers in contrast to the rows of other odd integer squares which follow the pentagonal numbers [5].

For $c^2 - a^2 = b^2$, $c \in 4r_1 + 1$, $a \in 4r_1' + 1$, $b \in 4r_0$

Thus

$$r_1 - r_1' = r_0 \tag{3.3}$$

Class of a^2,c^2	**No.**	$(r_1 - r_1')$	**Parity of**	
			n	n'
$\bar{1}_4$	1	$3(n - n')(3(n + n') + 1)$	even	even
$\bar{1}_4$	2	$3(n + n')(3(n - n') + 1)$	even	odd
$\bar{1}_4$	3	$3(n + n')(3(n - n') - 1)$	odd	even
$\bar{1}_4$	4	$3(n - n')(3(n + n') - 1)$	odd	odd

Table 5

If neither *a* nor *b* has 3 as a factor, then the rows r_1, r_1', r_0 will follow the pentagonal numbers $\left(\frac{1}{2}n(3n \pm 1)\right)$. Thus possible *f*(*n*) for $(r_1 - r_1')$ are listed in Table 5, and for r_0 in Table 6. As shown previously [5], even **b** can never be in Class $\bar{2}_4$ and r_0 does not have a factor 3 (Table 6), and so solutions are not possible since $3 \mid r_1 - r_1'$.

Class of b^2	**No.**	r_0
$\bar{0}_4$	1	2^q
$\bar{0}_4$	2	$2^q(12n(3n + 1) + 1)$
$\bar{0}_4$	3	$2^q(12n(3n - 1) + 1)$

Table 6

4. Concluding Comments

Topics for further research could be to investigate if other polynomials occur with other moduli under suitable divisibility non-divisibility conditions. Relevant number theoretic ideas may be found in [2, 9, 10].

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