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# An improved solution of $\sum_{i=1}^{k} \frac{1}{Xi} = 1$ in distinct integers when $x_i \nmid x_j$ for $i \neq j$

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Dedicated to the Memory of my parents Lucia and Aharon

**Abstract**

An improved solution of the title equation with k = 52, $x_{52} = 1963$ is exhibited. This is the best known result thus far.

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**Keywords.** Diophantine equations, Egyptian fractions.

## 1. Introduction

The diophantine equation in integers

$$\sum_{i=1}^{k} \frac{1}{Xi} = 1, \qquad x_1 < x_2 < \dots < x_k, \qquad x_i \nmid x_j \quad \text{for} \quad i \neq j \tag{1}$$

was considered by the late Paul Erdös, R.L.Graham, E.J. Barbeau, the author and others.

The existence of a solution of (1) was independently raised by the author [2,3] and R.L.Graham [6,3]. The author [3] provided a solution of (1), for which he received a reward of $10 offered by Erdös [2,3]. In this solution k = 79, but not all $x_i$ are products of two distinct primes. In [4], the author exhibited an example of the same nature as that in [3] with k = 68. Barbeau [1] gave an example with the stronger condition that each $x_i$ is a product of two distinct primes using 101 fractions and $x_{101} = 1838171$.

The author [5] improved Barbeau's solution with $k = 63$ and $x_{63} = 7909$, where each $x_i$ is a product of two distinct primes. The results of Barbeau [1] and of the author [3] are mentioned in [7].

In Section 2, we will demonstrate the main result of this paper, namely Solution 1.

## 2. The improved solution

**Solution 1.** The 52 different numbers

| | | | | | |
|---|---|---|---|---|---|
| 6 | 10 | 14 | 15 | 21 | 35 |
| 22 | 33 | 55 | 77 | | |
| 26 | 39 | 65 | 91 | | |
| 34 | 51 | 119 | 187 | | |
| 38 | 57 | 95 | 133 | | |
| 46 | 69 | 161 | 299 | | |
| 58 | 87 | 203 | 319 | | |
| 62 | 93 | 155 | | | |
| 82 | 123 | 287 | | | |
| 106 | 159 | 265 | 583 | | |
| 122 | 183 | 671 | | | |
| 213 | 355 | 497 | | | |
| 202 | 505 | 1313 | | | |
| 453 | 1057 | 1963 | | | |

have the following two properties:

(i) the sum of their reciprocals is equal to 1,

(ii) no-one divides any other.

**Proof**. (i) Solution 1 is arranged in fourteen rows. We compute

$$\sum_{i=1}^{52} \frac{1}{Xi} = 1$$

without using a computer as follows:

The least common multiple, in short $L$ of the first two rows is clearly $L = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. The numbers in these rows are all the products of two different factors of $L$.

The structure of each of the remaining twelve rows is as follows. The members of each row are of the form $Mp$ where $p \geq 13$ is a prime and all twelve primes are distinct. The values $M$ in rows 6, 13 and 14 consist of one prime factor of $13L$, and in the rest of the rows of one prime factor of $L$.

Computing the sum of the reciprocals of the numbers in any of the last twelve rows, one obtains a fraction whose denominator is clearly a multiple of $p$, and it comes out that the numerator must also be a multiple of $p$. After simplification by $p$, the denominator becomes a divisor of $13L$.

This enables us to carry out the summation without the aid of a computer. The sum of the fourteen partial sums adds up to 1. □

(ii) All 52 numbers in Solution 1 are of the form $pq$, where $p$, $q$ are distinct primes. Therefore, no-one divides any other. □

Solution 1 improves all known former results. It is composed of all the smallest possible parameters. Compared with [5], these are as follows:

**1.** k = 52 (k = 63).

**2.** $x_{52} = 1963$ ($x_{63} = 7909$).

**3.** there occur 17 primes (21 primes).

**4.** the largest prime is $p = 151$ ($p = 719$).

**Remark 1.** The 17 primes occurring in Solution 1 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 53, 61, 71, 101, 151, the first 11 of which are the smallest 11 primes.

**Remark 2.** One can show that no integer $x_t$ in (1) can be a power of a prime. Therefore, the smallest possible $x_1$ is 6.

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