NNTDM 15 (2009), 3, 1-8

On the Euler minimum and maximum functions

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1. The **Euler minimum function** is defined by

$$E(n) = \min\{k \geq 1 : \ n|\varphi(k)\} \tag{1}$$

It was introduced by P. Moree and H. Roskam [5]; and independently by J. Sándor [7], as a particular case of the more general function

$$F_f^A(n) = \min\{k \in A : \ n|f(k)\} \quad (A \subset \mathbb{N}^*), \tag{2}$$

where $f : \mathbb{N}^* \to \mathbb{N}^*$ is a given function, and A is a given set of positive integers. For $A = \mathbb{N}^*$, $f = \varphi$ (Euler's totient), one obtains the function E given by (1) (denoted also as F_φ in [7]). Since by Dirichlet's theorem on arithmetical progression, there exists $a \geq 1$ such that $k = an + 1 =$ prime, by $\varphi(k) = an \vdots n$, so $E(n)$ is well defined.

We note that for $A = \mathbb{N}^*$, $f(k) = k!$ one reobtains the Smarandache function

$$S(n) = \min\{k \geq 1 : \ n|k!\}, \tag{3}$$

while for $A = P = \{2, 3, 5, \dots\}$ = set of all primes, $f(k) = k!$, (2) gives a new function, denoted by us as $P(n)$:

$$P(n) = \min\{k \in P : \ n|k!\} \tag{4}$$

We note that this function should not be confused with the greatest prime divisor of n (denoted also sometimes by $P(n)$). For properties of this function, see [6], [7].

There is a **dual** of (2) (see [7]), namely

$$G_g^A(n) = \max\{k \in A : \ g(k)|n\}, \tag{5}$$

where $g : \mathbb{N}^* \to \mathbb{N}^*$, $A \subset \mathbb{N}^*$ are given, if this is well defined. For $A = \mathbb{N}^*$, $g(k) = k!$, this has been denoted by us by $S_*(n)$, and called as the dual of the Smarandache function:

$$S_*(n) = \max\{k \geq 1 : \ k!|n\} \tag{6}$$

For properties of this function, see [6]. See also F. Luca [4], where a conjecture of the author has been proved, and M. Le [3] for a recent new proof. See also K. Atanassov [1].

For $A = \mathbb{N}^*$, $g(k) = \varphi(k)$ one obtains the dual $E_*(n)$ of the Euler minimum function, which we shall call as the **Euler maximum function**:

$$E_*(n) = \max\{k \geq 1 : \ \varphi(k)|n\} \tag{7}$$

Since for $k > 6$, $\varphi(k) > \sqrt{k}$, clearly $k < n^2$, so $E_*(n) \leq n^2 < \infty$.

Generally, for $A = \mathbb{N}^*$, let us write simply $F_f^A(n) = F_f(n)$, $G_g^A(n) = G_f(n)$.

2. First we prove the following property of the Euler minimum function:

Theorem 1. *If p_i $(i = \overline{1,r})$ are distinct primes, and $\alpha_i \geq 1$ are integers, then*

$$\max\{E(p_i^{\alpha_i}) : \ i = \overline{1,r}\} \leq E\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leq [E(p_1^{\alpha_1}), \ldots, E(p_r^{\alpha_r})], \tag{8}$$

where $[, \ldots,]$ denotes l.c.m.

Proof. For simplicity we shall prove (8) for $r = 2$. Let p^α, q^β be distinct prime powers. Then $E(p^\alpha q^\beta) = \min\{k \geq 1 :\ p^\alpha q^\beta | \varphi(k)\} = k_0$, so $p^\alpha q^\beta | \varphi(k_0)$, which is equivalent to $p^\alpha | \varphi(k_0)$, $q^\beta | \varphi(k_0)$, thus $k_0 \geq E(p^\alpha)$, $k_0 \geq E(q^\beta)$, implying $E(p^\alpha q^\beta) \geq \max\{E(p^\alpha), E(q^\beta)\}$. It is immediate that the same proof applies to $E\left(\prod p^\alpha\right) \geq \max\{E(p^\alpha)\}$, where p^α are distinct prime powers. Therefore, the left side of (8) follows.

Now, let $E(p^\alpha) = k_1$, $E(q^\beta) = k_2$, implying $p^\alpha | \varphi(k_1)$, $q^\beta | \varphi(k_2)$. Let $[k_1, k_2] = g$. Since $k_1 | g$, one has $\varphi(k_1) | \varphi(g)$ (by a known property of the function φ). Similarly, since $k_2 | g$, one can write $\varphi(k_2) | \varphi(g)$. Thus $p^\alpha | \varphi(k_1) | \varphi(g)$ and $q^\beta | \varphi(k_2) | \varphi(g)$, yielding $p^\alpha q^\beta | \varphi(g)$. By the definition (1) this gives $g \geq E(p^\alpha q^\beta)$, i.e. $[E(p^\alpha), E(q^\beta)] \geq E(p^\alpha q^\beta)$, so the right side of (8) for $r = 2$ is proved. The general case follows exactly the same lines.

Remark 1. The above proof shows that the left side of (8) holds true for any function f (for which F_f is well defined), so we get

$$\max\{F_f(p_i^{\alpha_i}) :\ i = \overline{1, r}\} \leq F_f\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \tag{9}$$

For the right side of (8), with the same proof the following is valid: if f has the property

$$a | b \ \Rightarrow\ f(a) | f(b) \quad (a, b \geq 1), \tag{10}$$

then

$$F_f\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leq [F_f(p_1^{\alpha_1}), \ldots, F_f(p_r^{\alpha_r})] \tag{11}$$

Now, if one replaces (10) with a stronger property, then a better result will be true:

Theorem 2. *Assume that f satisfies the following property*

$$a \leq b \ \Rightarrow\ f(a) | f(b) \quad (a, b \geq 1) \tag{12}$$

Then

$$F_f\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) = \max\{F_f(p_i^{\alpha_i}):\ i=\overline{1,r}\} \tag{13}$$

Proof. By taking into account of (9), one needs only to show that the reverse inequality is true. For simplicity, let us take again $r=2$. Let $F_f(p^\alpha)=m$, $F_f(q^\beta)=n$, with $m \le n$. Then the definition (2) of F_f implies that $p^\alpha|f(m)$, $q^\beta|f(n)$. By (12) one has $f(m)|f(n)$, so $p^\alpha|f(m)|f(n)$. We have $p^\alpha|f(n)$, $q^\beta|f(n)$, so $p^\alpha q^\beta|f(n)$. But this implies $n \ge F_f(p^\alpha q^\beta)$, i.e. $\max\{F_f(p^\alpha), F_f(q^\beta)\} \ge F_f(p^\alpha q^\beta)$. The general case follows exactly the same lines.

Remark 2. If $(a,b)=1$, by writing $a=\prod_{i=1}^{r} p_i^{\alpha_i}$, $b=\prod_{j=1}^{s} q_j^{\beta_j}$, $(p_i,q_j)=1$, it follows that

$$F_f(ab) = \max\{E(p_i^{\alpha_i}), E(q_j^{\beta_j}):\ i=\overline{1,r},\ j=\overline{1,s}\} =$$

$$= \max\{\max\{E(p_i^{\alpha_i}):\ i=\overline{1,r}\}, \max\{E(q_j^{\beta_j}):\ j=\overline{1,s}\}\} =$$

$$= \max\{F_f(a), F_f(b)\},$$

so:

$$F_f(ab) = \max\{F_f(a), F_f(b)\} \text{ for } (a,b)=1 \tag{14}$$

When $f(n)=n!$, then clearly (12) is true, so (14) gives:

$$S(ab) = \max\{S(a), S(b)\} \text{ for } (a,b)=1, \tag{15}$$

discovered by F. Smarandache [9].

3. The Euler minimum function must be studied essentially (by Theorem 1) for prime powers p^α. For values of $E(p)$, $E(p^2)$, etc., see [5]. On the other hand, for each prime $p \ge 3$ one has

$$E(p-1) = p \tag{16}$$

Indeed, if $(p-1)|\varphi(k)$, then $p-1 \leq \varphi(k)$. Since $\varphi(k) \leq k-1$ for $k \geq 3$, one has $p \leq k$. Now $k = p$ gives $\varphi(p) = p-1$, giving (16).

The values of the Euler maximum function E_* given by (7) however are even difficult to calculate in some cases. This function doesn't seem to have been studied up to now.

Clearly $E_*(1) = 2$ since $\varphi(1) = 1$, $\varphi(2) = 1$. One has $E_*(2) = 6$ since $\varphi(6) = 2$, $2|2$, and it is well-known that $\varphi(n) \geq 4$ for $n \geq 7$. Now let $p \geq 3$ be a prime. Since $\varphi(k)|p$ implies $\varphi(k) = 1$ or $\varphi(k) = p$, for $p \geq 3$ the last equality is impossible for $\varphi(k)$ is even for all $k \geq 3$, we can have only $\varphi(k) = 1$ and $k_{max} = 2$. Actually since for $k \geq 3$, $\varphi(k)$ is even, $\varphi(k)|n$ is impossible for $n =$ odd, so remains $k \leq 2$, and $k_{max} = 2$. We have proved:

Theorem 3. *One has* $E_*(1) = 2$*,* $E_*(p) = \begin{cases} 6, & \textit{if} \ \ p = 2 \\ 2, & \textit{if} \ \ p \geq 3 \end{cases}$ *for all primes* p*; and* $E_*(n) = 2$ *for all* $n \geq 1$ *odd. For all* $n \geq 2$ *one has* $E_*(n) \geq 2$. (17)

The last inequality is a consequence of $\varphi(2) = 1$ and the definition (7).

The value 2 is taken infinitely often, but the same is true for the value 6:

Theorem 4. *For all* $\alpha \geq 1$ *one has*

$$E_*(2 \cdot 7^\alpha) = 6 \tag{18}$$

Proof. If $\varphi(k)|(2 \cdot 7^\alpha)$, then assuming $k \geq 3$, as $\varphi(k)$ is even, we can only have $\varphi(k) = 2$ or $\varphi(k) = 2 \cdot 7^a$ where $1 \leq a \leq \alpha$. Now, A. Schinzel [8] has shown that the equation $\varphi(x) = 2 \cdot 7^a$ is not solvable for any $a \geq 1$. Thus, it remains $\varphi(k) = 2$ and the maximal value of $k \geq 3$ is $k = 6$. This finishes the proof of (18).

Remark. One has similarly $E_*(2 \cdot 5^{2\alpha}) = 6$ for any $\alpha \geq 1$. (19)

The function E_* can take greater values, too; the values at powers of 2 is shown by the following theorem:

Theorem 5. *$E_*(2^m) = k$, where k is the greatest number which can be written as $k = 2^\alpha p_1 \dots p_r$, with $p_1 = 2^{2^{\alpha_1}}+1, \dots, p_r = 2^{2^{\alpha_r}}+1$ distinct Fermat primes, and where $\alpha = a+1-(2^{k_1}+\dots+2^{k_r})$, with $k_1, \dots, k_r \geq 0$, $0 \leq a \leq m$.* (20)

Proof. Since $\varphi(k)|2^m$, clearly $\varphi(k) = 2^a$, where $0 \leq a \leq m$. Now let $k = 2^\alpha p_1^{\alpha_1} \dots p_r^{\alpha_r}$ with $p_1, \dots, p_r$ distinct odd primes. Since $\varphi(k) = 2^{\alpha-1} p_1^{\alpha_1-1} \dots p_r^{\alpha_r-1}(p_1-1)\dots(p_r-1) = 2^a$, we must have $\alpha_1 - 1 = \dots = \alpha_r - 1 = 0$ and $p_1 - 1 = 2^{a_1}, \dots, p_r - 1 = 2^{a_r}$ with $\alpha - 1 + a_1 + \dots + a_r = a$. This gives $p_1 = 2^{a_1}+1, \dots, p_r = 2^{a_r}+1$. Since p_1 is prime, it is well-known that it is a Fermat prime, so $a_1 = 2^{k_1}$, etc., and the theorem follows.

Remark 3. For $m = 2$ we get $\alpha \leq 3 - (2^{k_1} + \dots + 2^{k_r})$, so with $k_1 = 0$ (when $p_1 = 3$), we get $k = 2^2 \cdot 3 = 12$. Another value would be $k = 2 \cdot 5 = 10$, so we get

$$E_*(4) = 12$$

Similarly, for $m = 3$,

$$E_*(8) = 30$$

If can be shown also that

$$E_*(16) = 60, \quad E_*(32) = 120, \quad E_*(64) = 240, \quad E_*(128) = 510, \text{ etc.}$$

However, since the structure (or the cardinality) of the Fermat primes is not well-known, there are problems also with the calculation of great values of $E_*(2^m)$.

The function $E_*(n)$ can take arbitrarily large values, since one has:

Theorem 6. *For all $m \geq 1$ the following inequality is true:*

$$E_*(m!) \geq \frac{(m!)^2}{\varphi(m!)} \tag{21}$$

Proof. It is known (see e.g. [2]) that the equation

$$\varphi(x) = m! \tag{22}$$

admits the solution $x = (m!)^2/\varphi(m!)$. Now, since $\varphi(x) = m!|m!$, clearly $E_*(m!) \geq x$, giving inequality (21).

Corollary. $\lim_{m\to\infty} \dfrac{E_*(m!)}{m!} = +\infty \qquad (23)$

Proof. Indeed, it is well-known (see e.g. [10]) that $\dfrac{m!}{\varphi(m!)} \to \infty$ as $m \to \infty$. By (21), this implies (23).

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