

A remark on ψ -function and Fibonacci sequence

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1 Introduction

The well-known Fibonacci sequence

$$F(0) = 0, F(1) = 1,$$

$$F(n+2) = F(n+1) + F(n), \text{ for every natural number } n \geq 0$$

has an extension with the form:

$$F(0) = a, F(1) = b,$$

$$F(n+2) = pF(n+1) + qF(n), \text{ for every natural number } n \geq 0,$$

where a, b, p, q are fixed natural numbers.

A property of this sequence will be discuss below.

2 Remarks on ψ -function

A digital arithmetic function will be described, following [1, 2, 3].

Let

$$n = \sum_{i=1}^k a_i \cdot 10^{k-i} \equiv \overline{a_1 a_2 \dots a_k},$$

where a_i is a natural number and $0 \leq a_i \leq 9$ ($1 \leq i \leq k$). Let for $n = 0 : \varphi(n) = 0$ and for $n > 0$:

$$\varphi(n) = \sum_{i=1}^k a_i.$$

We shall use the decimal count system everywhere hereafter.

Let us define a sequence of functions $\varphi_0, \varphi_1, \varphi_2, \dots$, where (l is a natural number)

$$\varphi_0(n) = n,$$

$$\varphi_{l+1} = \varphi(\varphi_l(n)).$$

Since for $k > 1$

$$\varphi(n) = \sum_{i=1}^k a_i < \sum_{i=1}^k a_i \cdot 10^{k-i} = n.$$

Then for every natural number n , the natural number l will exist so that

$$\varphi_l(n) = \varphi_{l+1}(n) \in \Delta \equiv \{0, 1, 2, \dots, 9\}.$$

Let function ψ be defined by

$$\psi(n) = \varphi_l(n),$$

where

$$\varphi_{l+1}(n) = \varphi_l(n).$$

In [1] the following two equalities are proved for each natural numbers m, n :

$$\psi(m + 9n) = \psi(m),$$

$$\psi(m^7) = \psi(m).$$

Let be given the sequence a_1, a_2, \dots , with its members being natural numbers and let

$$c_i = \psi(a_i) \quad (i = 1, 2, \dots).$$

Hence, we deduce the sequence c_1, c_2, \dots from the former sequence. If k and l exist so that $l \geq 0$,

$$c_{i+l} = c_{k+i+l} = c_{2k+i+l} = \dots$$

for $1 \leq i \leq k$, then we, following [1], shall say that

$$[c_{l+1}, c_{l+2}, \dots, c_{l+k}]$$

is *base* of the sequence a_1, a_2, \dots with length of k and with respect to function ψ .

For example, in [1] it is shown that ordinary Fibonacci sequence has base of length 24 with respect to the function ψ and is the following:

$$[1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9];$$

Lucas sequence also has a base of length 24 with respect to the function ψ and it is the following:

$$[2, 1, 3, 4, 7, 2, 9, 2, 2, 4, 6, 1, 7, 8, 6, 5, 2, 7, 9, 7, 7, 5, 3, 8],$$

while Lucas-Lehmer sequence $\{l_i\}_{i=0}^{\infty}$, for which

$$l_1 = 4, l_{n+1} = l_n^2 - 2 \quad (n \geq 0)$$

has a base [5] (i.e., with length 1) with respect to the function ψ . In [2] the authors proved that Euler's sequence, defined by

$$E_n = 2^n E_n\left(\frac{1}{2}\right)$$

has a base [1, 5, 7, 8, 4, 2], where the generating function for the Euler's polynomials $E_n(x)$ is defined by

$$\frac{e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \cdot \frac{t^n}{n!}$$

for $|t| < \pi$.

3 Main results

Theorem The base of extended Fibonacci sequence has the following members with respect to ψ -function:

- 1 $\psi(qa + pb)$
- 2 $\psi(pqa + (p^2 + q)b)$
- 3 $\psi((p^2q + q^2)a + (p^3 + 2pq)b)$
- 4 $\psi((p^3q + 2pq^2)a + (p^4 + 3p^2q + q^2)b)$
- 5 $\psi(((p^4q + 3p^2q^2 + q^3)a + (p^5 + 4p^3q + 3pq^2)b)$
- 6 $\psi((p^5q + 4p^3q^2 + 3pq^3)a + (p^6 + 5p^4q + 6p^2q^2 + q^3)b)$
- 7 $\psi((p^6q + 5p^4q^2 + 6p^2q^3 + q^4)a + (6p^5q + p^3q^2 + 4pq^3 + p)b)$
- 8 $\psi((6p^5q^2 + p^3q^3 + 4pq^4 + pq)a + (7p^6q + 6p^4q^2 + p^2q^3 + p^2 + q^4)b)$
- 9 $\psi((7p^6q^2 + 6p^4q^3 + p^2q^4 + p^2q + q^5)a + (3p^5q^2 + 2p^3q^3 + p^3 + 5pq^4 + 8pq)b)$
- 10 $\psi((3p^5q^3 + 2p^3q^4 + p^3q + 5pq^5 + 8pq^2)a + (p^6q^2 + 8p^4q^3 + p^4 + 6p^2q^4 + q^5)b)$
- 11 $\psi((p^6q^3 + 8p^4q^4 + p^4q + 6p^2q^5 + q^6)a + (2p^5q^3 + p^5 + 8p^3q^4 + p^3q + 6pq^5)b)$
- 12 $\psi((2p^5q^4 + p^5q + 8p^3q^5 + p^3q^2 + 6pq^6)a + (3p^6q^3 + p^6 + 7p^4q^4 + 2p^4q + 3p^2q^5 + q^6)b)$
- 13 $\psi((3p^6q^4 + p^6q + 7p^4q^5 + 2p^4q^2 + 3p^2q^6 + q)a + (3p^5q + 2p^3q^5 + p^3q^2 + 7pq^6 + 3pq^3 + p)b)$
- 14 $\psi((3p^5q^2 + 2p^3q^6 + p^3q^3 + 3pq^4 + 8pq)a + (3p^6q^4 + 4p^6q + 3p^4q^2 + p^2q^6 + 3p^2q^3 + p^2 + q)b)$
- 15 $\psi((3p^6q^5 + 4p^6q^2 + 3p^4q^3 + 3p^2q^4 + 2p^2q + q^2)a + (6p^5q^2 + 3p^3q^6 + 4p^3q^3 + p^3 + 6pq^4 + 4pq)b)$
- 16 $\psi((6p^5q^3 + 4p^3q^4 + 4p^3q + 6pq^5 + 4pq^2)a + (3p^6q^5 + p^6q^2 + 3p^4q^6 + 7p^4q^3 + p^4 + 6p^2q + q^2)b)$
- 17 $\psi((3p^6q^6 + p^6q^3 + 7p^4q^4 + 4p^4q + 6p^2q^2 + q^3)a + (3p^5q^6 + 4p^5q^3 + p^5 + 4p^3q^4 + p^3q + 6pq^2)b)$
- 18 $\psi((4p^5q^4 + 4p^5q + 4p^3q^5 + p^3q^2 + 6pq^3)a + (6p^6q^6 + 5p^6q^3 + p^6 + 2p^4q^4 + 5p^4q + 3p^2q^2 + q^3)b)$
- 19 $\psi((5p^6q^4 + 7p^6q + 2p^4q^5 + 5p^4q^2 + 3p^2q^3 + q^4)a + (6p^5q^4 + 4p^3q^5 + 4p^3q^2 + 6pq^6 + 3pq^3 + p)b)$
- 20 $\psi((6p^5q^5 + 4p^3q^6 + 4p^3q^3 + 3pq^4 + 7pq)a + (2p^6q^4 + 7p^6q + 6p^4q^5 + 6p^2q^6 + 6p^2q^3 + p^2 + q^4)b)$
- 21 $\psi((2p^6q^5 + 7p^6q^2 + 6p^4q^6 + 6p^2q^4 + 7p^2q + q^5)a + (3p^5q^5 + p^3q^6 + p^3q^3 + p^3 + 6pq^4 + 5pq)b)$
- 22 $\psi((3p^5q^6 + p^3q^4 + 2p^3q + 6pq^5 + 5pq^2)a + (5p^6q^5 + 7p^6q^2 + 7p^4q^6 + p^4q^3 + p^4 + 3p^2q^4 + 3p^2q + q^5)b)$
- 23 $\psi((5p^6q^6 + 7p^6q^3 + p^4q^4 + 8p^4q + 3p^2q^5 + 3p^2q^2 + q^6)a + (p^5q^6 + p^5q^3 + p^5 + 4p^3q^4 + 5p^3q + 3pq^5 + 3pq^2)b)$
- 24 $\psi((p^5q^4 + 2p^5q + 4p^3q^5 + 5p^3q^2 + 3pq^6 + 3pq^3)a + (6p^6q^6 + 8p^6q^3 + p^6 + 5p^4q^4 + 4p^4q + 6p^2q^5 + 6p^2q^2 + q^6)b)$
- 25 $\psi((8p^6q^4 + 7p^6q + 5p^4q^5 + 4p^4q^2 + 6p^2q^6 + 6p^2q^3 + q)a + (6p^5q^4 + 6p^5q + p^3q^5 + 2p^3q^2 + pq^6 + 2pq^3 + p)b)$
- 26 $\psi((6p^5q^5 + 6p^5q^2 + p^3q^6 + 2p^3q^3 + 2pq^4 + 2pq)a + (5p^6q^4 + 4p^6q + 6p^4q^5 + 6p^4q^2 + 7p^2q^6 + 8p^2q^3 + p^2 + q)b)$

- 27 $\psi((5p^6q^5 + 4p^6q^2 + 6p^4q^6 + 6p^4q^3 + 8p^2q^4 + 8p^2q + q^2)a + (3p^5q^5 + 3p^5q^2 + 8p^3q^6 + p^3q^3 + p^3 + 7pq^4 + 7pq)b)$
- 28 $\psi((3p^5q^6 + 3p^5q^3 + p^3q^4 + 7pq^5 + 7pq^2)a + (8p^6q^5 + 7p^6q^2 + 5p^4q^6 + 7p^4q^3 + p^4 + 6p^2q^4 + 6p^2q + q^2)b)$
- 29 $\psi((8p^6q^6 + 7p^6q^3 + 7p^4q^4 + 6p^4q + 6p^2q^5 + 6p^2q^2 + q^3)a + (8p^5q^6 + p^5q^3 + p^5 + 7p^3q^4 + 6p^3q + 6pq^5 + 6pq^2)b)$
- 30 $\psi((p^5q^4 + 7p^3q^5 + 6p^3q^2 + 6pq^6 + 6pq^3)a + (7p^6q^6 + 8p^6q^3 + p^6 + 5p^4q^4 + 3p^4q + 3p^2q^5 + 3p^2q^2 + q^3)b)$
- 31 $\psi((8p^6q^4 + 8p^6q + 5p^4q^5 + 3p^4q^2 + 3p^2q^6 + 3p^2q^3 + q^4)a + (6p^5q^4 + 3p^5q + p^3q^5 + 4pq^6 + 6pq^3 + p)b)$
- 32 $\psi((6p^5q^5 + 3p^5q^2 + p^3q^6 + 6pq^4 + 5pq)a + (5p^6q^4 + 2p^6q + 6p^4q^5 + 3p^4q^2 + 7p^2q^6 + p^2 + q^4)b)$
- 33 $\psi((5p^6q^5 + 2p^6q^2 + 6p^4q^6 + 3p^4q^3 + 8p^2q + q^5)a + (3p^5q^5 + 6p^5q^2 + 8p^3q^6 + p^3 + 3pq^4 + 7pq)b)$
- 34 $\psi((3p^5q^6 + 6p^5q^3 + 3pq^5 + 7pq^2)a + (8p^6q^5 + 8p^6q^2 + 5p^4q^6 + 3p^4q^3 + p^4 + 3p^2q^4 + 6p^2q + q^5)b)$
- 35 $\psi((8p^6q^6 + 8p^6q^3 + 3p^4q^4 + 6p^4q + 3p^2q^5 + 6p^2q^2 + q^6)a + (8p^5q^6 + p^5 + 3p^3q^4 + 6p^3q + 3pq^5 + 6pq^2)b)$
- 36 $\psi((3p^3q^5 + 6p^3q^2 + 3pq^6 + 6pq^3)a + (7p^6q^6 + 8p^6q^3 + p^6 + 6p^4q^4 + 3p^4q + 6p^2q^5 + 3p^2q^2 + q^6)b)$
- 37 $\psi((8p^6q^4 + 8p^6q + 6p^4q^5 + 3p^4q^2 + 6p^2q^6 + 3p^2q^3 + q)a + (6p^5q^4 + 3p^5q + 2pq^6 + 5pq^3 + p)b)$
- 38 $\psi((6p^5q^5 + 3p^5q^2 + 5pq^4 + 3pq)a + (5p^6q^4 + 2p^6q + 6p^4q^5 + 3p^4q^2 + 8p^2q^6 + 8p^2q^3 + p^2 + q)b)$
- 39 $\psi((5p^6q^5 + 2p^6q^2 + 6p^4q^6 + 3p^4q^3 + 8p^2q^4 + q^2)a + (3p^5q^5 + 6p^5q^2 + 8p^3q^6 + 8p^3q^3 + p^3 + pq^4 + 6pq)b)$
- 40 $\psi((3p^5q^6 + 6p^5q^3 + 8p^3q^4 + pq^5 + 6pq^2)a + (8p^6q^5 + 8p^6q^2 + 5p^4q^6 + 2p^4q^3 + p^4 + 6p^2q + q^2)b)$
- 41 $\psi((8p^6q^6 + 8p^6q^3 + 2p^4q^4 + 6p^4q + 6p^2q^2 + q^3)a + (8p^5q^6 + 8p^5q^3 + p^5 + 8p^3q^4 + 6p^3q + 6pq^2)b)$
- 42 $\psi((8p^5q^4 + 8p^3q^5 + 6p^3q^2 + 6pq^3)a + (7p^6q^6 + 7p^6q^3 + p^6 + p^4q^4 + 3p^4q + 3p^2q^2 + q^3)b)$
- 43 $\psi((7p^6q^4 + 8p^6q + p^4q^5 + 3p^4q^2 + 3p^2q^3 + q^4)a + (3p^5q + 8p^3q^5 + 7pq^6 + 5pq^3 + p)b)$
- 44 $\psi((3p^5q^2 + 8p^3q^6 + 5pq^4 + 8pq)a + (7p^6q^4 + 2p^6q + 3p^4q^2 + 7p^2q^6 + 8p^2q^3 + p^2 + q^4)b)$
- 45 $\psi((7p^6q^5 + 2p^6q^2 + 3p^4q^3 + 8p^2q^4 + 8p^2q + q^5)a + (6p^5q^2 + 6p^3q^6 + 8p^3q^3 + p^3 + 4pq^4 + pq)b)$
- 46 $\psi((6p^5q^3 + 8p^3q^4 + 7p^3q + 4pq^5 + pq^2)a + (7p^6q^5 + 8p^6q^2 + 6p^4q^6 + 2p^4q^3 + p^4 + 3p^2q^4 + q^5)b)$
- 47 $\psi((7p^6q^6 + 8p^6q^3 + 2p^4q^4 + 7p^4q + 3p^2q^5 + q^6)a + (6p^5q^6 + 8p^5q^3 + p^5 + 2p^3q^4 + 7p^3q + 3pq^5)b)$
- 48 $\psi((8p^5q^4 + 7p^5q + 2p^3q^5 + 7p^3q^2 + 3pq^6)a + (4p^6q^6 + 7p^6q^3 + p^6 + 4p^4q^4 + 5p^4q + 6p^2q^5 + q^6)b)$

- 49 $\psi((7p^6q^4 + 5p^6q + 4p^4q^5 + 5p^4q^2 + 6p^2q^6 + q)a + (3p^5q^4 + 3p^5q + 8p^3q^5 + 7p^3q^2 + 8pq^6 + 7pq^3 + p)b)$
- 50 $\psi((3p^5q^5 + 3p^5q^2 + 8p^3q^6 + 7p^3q^3 + 7pq^4)a + (p^6q^4 + 8p^6q + 3p^4q^5 + 3p^4q^2 + 5p^2q^6 + 7p^2q^3 + p^2 + q)b)$
- 51 $\psi((p^6q^5 + 8p^6q^2 + 3p^4q^6 + 3p^4q^3 + 7p^2q^4 + 6p^2q + q^2)a + (6p^5q^5 + 6p^5q^2 + 4p^3q^6 + 5p^3q^3 + p^3 + 8pq^4)b)$
- 52 $\psi((6p^5q^6 + 6p^5q^3 + 5p^3q^4 + 5p^3q + 8pq^5)a + (7p^6q^5 + 5p^6q^2 + 7p^4q^6 + 8p^4q^3 + p^4 + 6p^2q^4 + 6p^2q + q^2)b)$
- 53 $\psi((7p^6q^6 + 5p^6q^3 + 8p^4q^4 + 8p^4q + 6p^2q^5 + 6p^2q^2 + q^3)a + (4p^5q^6 + 5p^5q^3 + p^5 + 2p^3q^4 + 2p^3q + 6pq^5 + 6pq^2)b)$
- 54 $\psi((5p^5q^4 + 5p^5q + 2p^3q^5 + 2p^3q^2 + 6pq^6 + 6pq^3)a + (2p^6q^6 + p^6q^3 + p^6 + p^4q^4 + p^4q + 3p^2q^5 + 3p^2q^2 + q^3)b)$
- 55 $\psi((p^6q^4 + 3p^6q + p^4q^5 + p^4q^2 + 3p^2q^6 + 3p^2q^3 + q^4)a + (6p^5q^4 + 6p^5q + 5p^3q^5 + 5p^3q^2 + 8pq^6 + 8pq^3 + p)b)$
- 56 $\psi((6p^5q^5 + 6p^5q^2 + 5p^3q^6 + 5p^3q^3 + 8pq^4)a + (7p^6q^4 + 6p^4q^5 + 6p^4q^2 + 2p^2q^6 + 2p^2q^3 + p^2 + q^4)b)$
- 57 $\psi((7p^6q^5 + 6p^4q^6 + 6p^4q^3 + 2p^2q^4 + 3p^2q + q^5)a + (3p^5q^5 + 3p^5q^2 + 7p^3q^6 + 7p^3q^3 + p^3 + 7pq^4)b)$
- 58 $\psi((3p^5q^6 + 3p^5q^3 + 7p^3q^4 + 8p^3q + 7pq^5)a + (p^6q^5 + 3p^6q^2 + 4p^4q^6 + 4p^4q^3 + p^4 + 3p^2q + q^5)b)$
- 59 $\psi((p^6q^6 + 3p^6q^3 + 4p^4q^4 + 5p^4q + 3p^2q^2 + q^6)a + (7p^5q^6 + 7p^5q^3 + p^5 + 7p^3q^4 + 2p^3q + 3pq^2)b)$
- 60 $\psi((7p^5q^4 + 8p^5q + 7p^3q^5 + 2p^3q^2 + 3pq^3)a + (8p^6q^6 + p^6q^3 + p^6 + 2p^4q^4 + 7p^4q + 6p^2q^2 + q^6)b)$
- 61 $\psi((p^6q^4 + 2p^4q^5 + 7p^4q^2 + 6p^2q^3 + q)a + (6p^5q + 7p^3q^5 + 8p^3q^2 + 4pq^3 + p)b)$
- 62 $\psi((6p^5q^2 + 7p^3q^6 + 8p^3q^3 + 4pq^4 + pq)a + (p^6q^4 + 6p^6q + 6p^4q^2 + p^2q^3 + p^2 + q)b)$
- 63 $\psi((p^6q^5 + 6p^6q^2 + 6p^4q^3 + p^2q^4 + p^2q + q^2)a + (3p^5q^2 + 7p^3q^6 + p^3 + 5pq^4 + 8pq)b)$
- 64 $\psi((3p^5q^3 + 8p^3q + 5pq^5 + 8pq^2)a + (p^6q^5 + 7p^4q^6 + 6p^4q^3 + p^4 + 6p^2q^4 + q^2)b)$
- 65 $\psi((p^6q^6 + 6p^4q^4 + 8p^4q + 6p^2q^5 + q^3)a + (7p^5q^6 + p^5 + 6p^3q^4 + 8p^3q + 6pq^5)b)$
- 66 $\psi((8p^5q + 6p^3q^5 + 8p^3q^2 + 6pq^6)a + (8p^6q^6 + p^6 + 3p^4q^4 + 7p^4q + 3p^2q^5 + q^3)b)$
- 67 $\psi((3p^4q^5 + 7p^4q^2 + 3p^2q^6 + q^4)a + (3p^5q^4 + 6p^5q + 8p^3q^2 + 5pq^6 + pq^3 + p)b)$
- 68 $\psi((3p^5q^5 + 6p^5q^2 + 8p^3q^3 + pq^4 + 6pq)a + (3p^6q^4 + 6p^6q + 3p^4q^5 + 6p^4q^2 + 8p^2q^6 + p^2q^3 + p^2 + q^4)b)$
- 69 $\psi((3p^6q^5 + 6p^6q^2 + 3p^4q^6 + 6p^4q^3 + p^2q^4 + q^5)a + (6p^5q^5 + 3p^5q^2 + 8p^3q^6 + p^3 + 5pq^4 + 3pq)b)$
- 70 $\psi((6p^5q^6 + 3p^5q^3 + 5pq^5 + 3pq^2)a + (2p^4q^6 + 6p^4q^3 + p^4 + 6p^2q^4 + 3p^2q + q^5)b)$
- 71 $\psi((6p^4q^4 + 3p^4q + 6p^2q^5 + 3p^2q^2 + q^6)a + (8p^5q^6 + p^5 + 6p^3q^4 + 3p^3q + 6pq^5 + 3pq^2)b)$
- 72 $\psi((6p^3q^5 + 3p^3q^2 + 6pq^6 + 3pq^3)a + (8p^6q^6 + p^6 + 3p^4q^4 + 6p^4q + 3p^2q^5 + 6p^2q^2 + q^6)b)$
- 73 $\psi((3p^4q^5 + 6p^4q^2 + 3p^2q^6 + 6p^2q^3 + q)a + (3p^5q^4 + 6p^5q + 6pq^6 + 3pq^3 + p)b)$
- 74 $\psi((3p^5q^5 + 6p^5q^2 + 3pq^4 + 7pq)a + (3p^6q^4 + 6p^6q + 3p^4q^5 + 6p^4q^2 + p^2 + q)b)$
- 75 $\psi((3p^6q^5 + 6p^6q^2 + 3p^4q^6 + 6p^4q^3 + p^2q + q^2)a + (6p^5q^5 + 3p^5q^2 + p^3 + 6pq^4 + 5pq)b)$
- 76 $\psi((6p^5q^6 + 3p^5q^3 + p^3q + 6pq^5 + 5pq^2)a + (3p^4q^6 + 6p^4q^3 + p^4 + 6p^2q^4 + 6p^2q + q^2)b)$

- 77 $\psi((6p^4q^4 + 4p^4q + 6p^2q^5 + 6p^2q^2 + q^3)a + (p^5 + 6p^3q^4 + 7p^3q + 6pq^5 + 6pq^2)b)$
- 78 $\psi((p^5q + 6p^3q^5 + 7p^3q^2 + 6pq^6 + 6pq^3)a + (p^6 + 3p^4q^4 + 2p^4q + 3p^2q^5 + 3p^2q^2 + q^3)b)$
- 79 $\psi((p^6q + 3p^4q^5 + 2p^4q^2 + 3p^2q^6 + 3p^2q^3 + q^4)a + (3p^5q^4 + 3p^5q + p^3q^2 + 6pq^6 + 7pq^3 + p)b)$
- 80 $\psi((3p^5q^5 + 3p^5q^2 + p^3q^3 + 7pq^4 + 7pq)a + (3p^6q^4 + 4p^6q + 3p^4q^5 + 3p^4q^2 + p^2q^3 + p^2 + q^4)b)$
- 81 $\psi((3p^6q^5 + 4p^6q^2 + 3p^4q^6 + 3p^4q^3 + p^2q^4 + p^2q + q^5)a + (6p^5q^5 + 6p^5q^2 + 2p^3q^3 + p^3 + 2pq^4 + 2pq)b)$
- 82 $\psi((6p^5q^6 + 6p^5q^3 + 2p^3q^4 + p^3q + 2pq^5 + 2pq^2)a + (p^6q^2 + 3p^4q^6 + 5p^4q^3 + p^4 + 3p^2q^4 + 3p^2q + q^5)b)$
- 83 $\psi((p^6q^3 + 5p^4q^4 + 4p^4q + 3p^2q^5 + 3p^2q^2 + q^6)a + (2p^5q^3 + p^5 + 5p^3q^4 + 4p^3q + 3pq^5 + 3pq^2)b)$
- 84 $\psi((2p^5q^4 + p^5q + 5p^3q^5 + 4p^3q^2 + 3pq^6 + 3pq^3)a + (3p^6q^3 + p^6 + p^4q^4 + 8p^4q + 6p^2q^5 + 6p^2q^2 + q^6)b)$
- 85 $\psi((3p^6q^4 + p^6q + p^4q^5 + 8p^4q^2 + 6p^2q^6 + 6p^2q^3 + q)a + (3p^5q^4 + 2p^3q^5 + p^3q^2 + 4pq^6 + 6pq^3 + p)b)$
- 86 $\psi((3p^5q^5 + 2p^3q^6 + p^3q^3 + 6pq^4 + 5pq)a + (6p^6q^4 + p^6q + 3p^4q^5 + p^2q^6 + 3p^2q^3 + p^2 + q)b)$
- 87 $\psi((6p^6q^5 + p^6q^2 + 3p^4q^6 + 3p^2q^4 + 2p^2q + q^2)a + (6p^5q^5 + 3p^3q^6 + 4p^3q^3 + p^3 + 3pq^4 + 7pq)b)$
- 88 $\psi((6p^5q^6 + 4p^3q^4 + 4p^3q + 3pq^5 + 7pq^2)a + (3p^6q^5 + p^6q^2 + 6p^4q^6 + 4p^4q^3 + p^4 + 6p^2q^4 + q^2)b)$
- 89 $\psi((3p^6q^6 + p^6q^3 + 4p^4q^4 + 7p^4q + 6p^2q^5 + q^3)a + (3p^5q^6 + 4p^5q^3 + p^5 + p^3q^4 + 4p^3q + 6pq^5)b)$
- 90 $\psi((4p^5q^4 + 4p^5q + p^3q^5 + 4p^3q^2 + 6pq^6)a + (6p^6q^6 + 5p^6q^3 + p^6 + 5p^4q^4 + 2p^4q + 3p^2q^5 + q^3)b)$
- 91 $\psi((5p^6q^4 + 7p^6q + 5p^4q^5 + 2p^4q^2 + 3p^2q^6 + q^4)a + (6p^5q + 4p^3q^5 + 4p^3q^2 + 3pq^6 + 6pq^3 + p)b)$
- 92 $\psi((6p^5q^2 + 4p^3q^6 + 4p^3q^3 + 6pq^4 + 4pq)a + (5p^6q^4 + 4p^6q + 6p^4q^2 + 6p^2q^6 + 6p^2q^3 + p^2 + q^4)b)$
- 93 $\psi((5p^6q^5 + 4p^6q^2 + 6p^4q^3 + 6p^2q^4 + 7p^2q + q^5)a + (3p^5q^2 + p^3q^6 + p^3q^3 + p^3 + 3pq^4 + 8pq)b)$
- 94 $\psi((3p^5q^3 + p^3q^4 + 2p^3q + 3pq^5 + 8pq^2)a + (5p^6q^5 + 7p^6q^2 + p^4q^6 + 7p^4q^3 + p^4 + 6p^2q + q^5)b)$
- 95 $\psi((5p^6q^6 + 7p^6q^3 + 7p^4q^4 + 2p^4q + 6p^2q^2 + q^6)a + (p^5q^6 + p^5q^3 + p^5 + p^3q^4 + 8p^3q + 6pq^2)b)$
- 96 $\psi((p^5q^4 + 2p^5q + p^3q^5 + 8p^3q^2 + 6pq^3)a + (6p^6q^6 + 8p^6q^3 + p^6 + 8p^4q^4 + p^4q + 3p^2q^2 + q^6)b)$
- 97 $\psi((8p^6q^4 + 7p^6q + 8p^4q^5 + p^4q^2 + 3p^2q^3 + q)a + (3p^5q + p^3q^5 + 2p^3q^2 + 7pq^6 + 5pq^3 + p)b)$
- 98 $\psi((3p^5q^2 + p^3q^6 + 2p^3q^3 + 5pq^4 + 8pq)a + (8p^6q^4 + p^6q + 3p^4q^2 + 7p^2q^6 + 8p^2q^3 + p^2 + q)b)$
- 99 $\psi((8p^6q^5 + p^6q^2 + 3p^4q^3 + 8p^2q^4 + 8p^2q + q^2)a + (6p^5q^2 + 8p^3q^6 + p^3q^3 + p^3 + 4pq^4 + pq)b)$

- 100 $\psi((6p^5q^3 + p^3q^4 + 4pq^5 + pq^2)a + (8p^6q^5 + 7p^6q^2 + 8p^4q^6 + 4p^4q^3 + p^4 + 3p^2q^4 + q^2)b)$
- 101 $\psi((8p^6q^6 + 7p^6q^3 + 4p^4q^4 + 3p^2q^5 + q^3)a + (8p^5q^6 + p^5q^3 + p^5 + 4p^3q^4 + 3pq^5)b)$
- 102 $\psi((p^5q^4 + 4p^3q^5 + 3pq^6)a + (7p^6q^6 + 8p^6q^3 + p^6 + 8p^4q^4 + 6p^2q^5 + q^3)b)$
- 103 $\psi((8p^6q^4 + 8p^6q + 8p^4q^5 + 6p^2q^6 + q^4)a + (p^3q^5 + pq^6 + p)b)$
- 104 $\psi((p^3q^6 + 2pq)a + (8p^6q^4 + 8p^6q + 7p^2q^6 + p^2 + q^4)b)$
- 105 $\psi((8p^6q^5 + 8p^6q^2 + 8p^2q + q^5)a + (8p^3q^6 + p^3 + pq)b)$
- 106 $\psi(pq^2a + (8p^6q^5 + 8p^6q^2 + 8p^4q^6 + p^4 + q^5)b)$
- 107 $\psi((8p^6q^6 + 8p^6q^3 + q^6)a + (8p^5q^6 + p^5)b)$
- 108 $\psi((7p^6q^6 + 8p^6q^3 + p^6 + q^6)b)$
- 109 $\psi((8p^6q^4 + 8p^6q + q)a + (8pq^6 + 8pq^3 + p)b)$
- 110 $\psi(8pq^4a + (8p^6q^4 + 8p^6q + 8p^2q^6 + 8p^2q^3 + p^2 + q)b)$
- 111 $\psi((8p^6q^5 + 8p^6q^2 + 8p^2q^4 + q^2)a + (8p^3q^6 + 8p^3q^3 + p^3 + 7pq^4)b)$
- 112 $\psi((8p^3q^4 + 7pq^5)a + (8p^6q^5 + 8p^6q^2 + 8p^4q^6 + 8p^4q^3 + p^4 + 6p^2q^4 + q^2)b)$
- 113 $\psi((8p^6q^6 + 8p^6q^3 + 8p^4q^4 + 6p^2q^5 + q^3)a + (8p^5q^6 + 8p^5q^3 + p^5 + 5p^3q^4 + 6pq^5)b)$
- 114 $\psi((8p^5q^4 + 5p^3q^5 + 6pq^6)a + (7p^6q^6 + 7p^6q^3 + p^6 + 4p^4q^4 + 3p^2q^5 + q^3)b)$
- 115 $\psi((7p^6q^4 + 8p^6q + 4p^4q^5 + 3p^2q^6 + q^4)a + (3p^5q^4 + 8p^3q^5 + 4pq^6 + 8pq^3 + p)b)$
- 116 $\psi((3p^5q^5 + 8p^3q^6 + 8pq^4 + 5pq)a + (p^6q^4 + 8p^6q + 3p^4q^5 + 7p^2q^6 + 8p^2q^3 + p^2 + q^4)b)$
- 117 $\psi((p^6q^5 + 8p^6q^2 + 3p^4q^6 + 8p^2q^4 + 8p^2q + q^5)a + (6p^5q^5 + 6p^3q^6 + 8p^3q^3 + p^3 + pq^4 + 4pq)b)$
- 118 $\psi((6p^5q^6 + 8p^3q^4 + 7p^3q + pq^5 + 4pq^2)a + (7p^6q^5 + 8p^6q^2 + 8p^4q^3 + p^4 + 3p^2q + q^5)b)$
- 119 $\psi((7p^6q^6 + 8p^6q^3 + 8p^4q^4 + p^4q + 3p^2q^2 + q^6)a + (6p^5q^6 + 8p^5q^3 + p^5 + 8p^3q^4 + p^3q + 3pq^2)b)$
- 120 $\psi((8p^5q^4 + 7p^5q + 8p^3q^5 + p^3q^2 + 3pq^3)a + (4p^6q^6 + 7p^6q^3 + p^6 + 7p^4q^4 + 2p^4q + 6p^2q^2 + q^6)b)$
- 121 $\psi((7p^6q^4 + 5p^6q + 7p^4q^5 + 2p^4q^2 + 6p^2q^3 + q)a + (6p^5q^4 + 8p^3q^5 + 7p^3q^2 + 5pq^6 + pq^3 + p)b)$
- 122 $\psi((6p^5q^5 + 8p^3q^6 + 7p^3q^3 + pq^4 + 6pq)a + (4p^6q^4 + 5p^6q + 6p^4q^5 + 5p^2q^6 + 7p^2q^3 + p^2 + q)b)$
- 123 $\psi((4p^6q^5 + 5p^6q^2 + 6p^4q^6 + 7p^2q^4 + 6p^2q + q^2)a + (3p^5q^5 + 4p^3q^6 + 5p^3q^3 + p^3 + 5pq^4 + 3pq)b)$
- 124 $\psi((3p^5q^6 + 5p^3q^4 + 5p^3q + 5pq^5 + 3pq^2)a + (7p^6q^5 + 5p^6q^2 + p^4q^6 + 5p^4q^3 + p^4 + 3p^2q^4 + q^2)b)$
- 125 $\psi((7p^6q^6 + 5p^6q^3 + 5p^4q^4 + 2p^4q + 3p^2q^5 + q^3)a + (4p^5q^6 + 5p^5q^3 + p^5 + 8p^3q^4 + 5p^3q + 3pq^5)b)$
- 126 $\psi((5p^5q^4 + 5p^5q + 8p^3q^5 + 5p^3q^2 + 3pq^6)a + (2p^6q^6 + p^6q^3 + p^6 + 4p^4q^4 + 7p^4q + 6p^2q^5 + q^3)b)$
- 127 $\psi((p^6q^4 + 3p^6q + 4p^4q^5 + 7p^4q^2 + 6p^2q^6 + q^4)a + (3p^5q + 5p^3q^5 + 5p^3q^2 + 5pq^6 + 2pq^3 + p)b)$
- 128 $\psi((3p^5q^2 + 5p^3q^6 + 5p^3q^3 + 2pq^4 + 6pq)a + (p^6q^4 + 6p^6q + 3p^4q^2 + 2p^2q^6 + 2p^2q^3 + p^2 + q^4)b)$

- 129 $\psi((p^6q^5 + 6p^6q^2 + 3p^4q^3 + 2p^2q^4 + 3p^2q + q^5)a + (6p^5q^2 + 7p^3q^6 + 7p^3q^3 + p^3 + 4pq^4 + 3pq)b)$
- 130 $\psi((6p^5q^3 + 7p^3q^4 + 8p^3q + 4pq^5 + 3pq^2)a + (p^6q^5 + 3p^6q^2 + 7p^4q^6 + p^4q^3 + p^4 + 6p^2q^4 + 6p^2q + q^5)b)$
- 131 $\psi((p^6q^6 + 3p^6q^3 + p^4q^4 + 8p^4q + 6p^2q^5 + 6p^2q^2 + q^6)a + (7p^5q^6 + 7p^5q^3 + p^5 + 4p^3q^4 + 5p^3q + 6pq^5 + 6pq^2)b)$
- 132 $\psi((7p^5q^4 + 8p^5q + 4p^3q^5 + 5p^3q^2 + 6pq^6 + 6pq^3)a + (8p^6q^6 + p^6q^3 + p^6 + 5p^4q^4 + 4p^4q + 3p^2q^5 + 3p^2q^2 + q^6)b)$
- 133 $\psi((p^6q^4 + 5p^4q^5 + 4p^4q^2 + 3p^2q^6 + 3p^2q^3 + q)a + (3p^5q^4 + 3p^5q + 7p^3q^5 + 8p^3q^2 + 6pq^6 + 7pq^3 + p)b)$
- 134 $\psi((3p^5q^5 + 3p^5q^2 + 7p^3q^6 + 8p^3q^3 + 7pq^4 + 7pq)a + (4p^6q^4 + 3p^6q + 3p^4q^5 + 3p^4q^2 + p^2q^3 + p^2 + q)b)$
- 135 $\psi((4p^6q^5 + 3p^6q^2 + 3p^4q^6 + 3p^4q^3 + p^2q^4 + p^2q + q^2)a + (6p^5q^5 + 6p^5q^2 + 7p^3q^6 + p^3 + 2pq^4 + 2pq)b)$
- 136 $\psi((6p^5q^6 + 6p^5q^3 + 8p^3q + 2pq^5 + 2pq^2)a + (p^6q^5 + p^4q^6 + 3p^4q^3 + p^4 + 3p^2q^4 + 3p^2q + q^2)b)$
- 137 $\psi((p^6q^6 + 3p^4q^4 + 2p^4q + 3p^2q^5 + 3p^2q^2 + q^3)a + (7p^5q^6 + p^5 + 3p^3q^4 + 2p^3q + 3pq^5 + 3pq^2)b)$
- 138 $\psi((8p^5q + 3p^3q^5 + 2p^3q^2 + 3pq^6 + 3pq^3)a + (8p^6q^6 + p^6 + 6p^4q^4 + 4p^4q + 6p^2q^5 + 6p^2q^2 + q^3)b)$
- 139 $\psi((6p^4q^5 + 4p^4q^2 + 6p^2q^6 + 6p^2q^3 + q^4)a + (6p^5q^4 + 3p^5q + 8p^3q^2 + 2pq^6 + 4pq^3 + p)b)$
- 140 $\psi((6p^5q^5 + 3p^5q^2 + 8p^3q^3 + 4pq^4 + 3pq)a + (6p^6q^4 + 3p^6q + 6p^4q^5 + 3p^4q^2 + 8p^2q^6 + p^2q^3 + p^2 + q^4)b)$
- 141 $\psi((6p^6q^5 + 3p^6q^2 + 6p^4q^6 + 3p^4q^3 + p^2q^4 + q^5)a + (3p^5q^5 + 6p^5q^2 + 8p^3q^6 + p^3 + 2pq^4 + 6pq)b)$
- 142 $\psi((3p^5q^6 + 6p^5q^3 + 2pq^5 + 6pq^2)a + (5p^4q^6 + 3p^4q^3 + p^4 + 3p^2q^4 + 6p^2q + q^5)b)$
- 143 $\psi((3p^4q^4 + 6p^4q + 3p^2q^5 + 6p^2q^2 + q^6)a + (8p^5q^6 + p^5 + 3p^3q^4 + 6p^3q + 3pq^5 + 6pq^2)b)$
- 144 $\psi((3p^3q^5 + 6p^3q^2 + 3pq^6 + 6pq^3)a + (8p^6q^6 + p^6 + 6p^4q^4 + 3p^4q + 6p^2q^5 + 3p^2q^2 + q^6)b)$
- 145 $\psi((6p^4q^5 + 3p^4q^2 + 6p^2q^6 + 3p^2q^3 + q)a + (6p^5q^4 + 3p^5q + 3pq^6 + 6pq^3 + p)b)$
- 146 $\psi((6p^5q^5 + 3p^5q^2 + 6pq^4 + 4pq)a + (6p^6q^4 + 3p^6q + 6p^4q^5 + 3p^4q^2 + p^2 + q)b)$
- 147 $\psi((6p^6q^5 + 3p^6q^2 + 6p^4q^6 + 3p^4q^3 + p^2q + q^2)a + (3p^5q^5 + 6p^5q^2 + p^3 + 3pq^4 + 8pq)b)$
- 148 $\psi((3p^5q^6 + 6p^5q^3 + p^3q + 3pq^5 + 8pq^2)a + (6p^4q^6 + 3p^4q^3 + p^4 + 3p^2q^4 + q^2)b)$
- 149 $\psi((3p^4q^4 + 7p^4q + 3p^2q^5 + q^3)a + (p^5 + 3p^3q^4 + p^3q + 3pq^5)b)$
- 150 $\psi((p^5q + 3p^3q^5 + p^3q^2 + 3pq^6)a + (p^6 + 6p^4q^4 + 8p^4q + 6p^2q^5 + q^3)b)$
- 151 $\psi((p^6q + 6p^4q^5 + 8p^4q^2 + 6p^2q^6 + q^4)a + (6p^5q^4 + p^3q^2 + 3pq^6 + pq^3 + p)b)$
- 152 $\psi((6p^5q^5 + p^3q^3 + pq^4 + 4pq)a + (6p^6q^4 + p^6q + 6p^4q^5 + p^2q^3 + p^2 + q^4)b)$
- 153 $\psi((6p^6q^5 + p^6q^2 + 6p^4q^6 + p^2q^4 + p^2q + q^5)a + (3p^5q^5 + 2p^3q^3 + p^3 + 8pq^4 + 5pq)b)$
- 154 $\psi((3p^5q^6 + 2p^3q^4 + p^3q + 8pq^5 + 5pq^2)a + (p^6q^2 + 6p^4q^6 + 2p^4q^3 + p^4 + 6p^2q + q^5)b)$
- 155 $\psi((p^6q^3 + 2p^4q^4 + 7p^4q + 6p^2q^2 + q^6)a + (2p^5q^3 + p^5 + 2p^3q^4 + 7p^3q + 6pq^2)b)$
- 156 $\psi((2p^5q^4 + p^5q + 2p^3q^5 + 7p^3q^2 + 6pq^3)a + (3p^6q^3 + p^6 + 4p^4q^4 + 5p^4q + 3p^2q^2 + q^6)b)$

- 157 $\psi((3p^6q^4 + p^6q + 4p^4q^5 + 5p^4q^2 + 3p^2q^3 + q)a + (6p^5q^4 + 6p^5q + 2p^3q^5 + p^3q^2 + pq^6 + p)b)$
- 158 $\psi((6p^5q^5 + 6p^5q^2 + 2p^3q^6 + p^3q^3 + 2pq)a + (7p^6q + 6p^4q^5 + 6p^4q^2 + p^2q^6 + 3p^2q^3 + p^2 + q)b)$
- 159 $\psi((7p^6q^2 + 6p^4q^6 + 6p^4q^3 + 3p^2q^4 + 2p^2q + q^2)a + (3p^5q^5 + 3p^5q^2 + 3p^3q^6 + 4p^3q^3 + p^3 + pq)b)$
- 160 $\psi((3p^5q^6 + 3p^5q^3 + 4p^3q^4 + 4p^3q + pq^2)a + (3p^6q^5 + p^6q^2 + p^4q^3 + p^4 + 3p^2q^4 + 3p^2q + q^2)b)$
- 161 $\psi((3p^6q^6 + p^6q^3 + p^4q^4 + p^4q + 3p^2q^5 + 3p^2q^2 + q^3)a + (3p^5q^6 + 4p^5q^3 + p^5 + 7p^3q^4 + 7p^3q + 3pq^5 + 3pq^2)b)$
- 162 $\psi((4p^5q^4 + 4p^5q + 7p^3q^5 + 7p^3q^2 + 3pq^6 + 3pq^3)a + (6p^6q^6 + 5p^6q^3 + p^6 + 8p^4q^4 + 8p^4q + 6p^2q^5 + 6p^2q^2 + q^3)b)$
- 163 $\psi((5p^6q^4 + 7p^6q + 8p^4q^5 + 8p^4q^2 + 6p^2q^6 + 6p^2q^3 + q^4)a + (3p^5q^4 + 3p^5q + 4p^3q^5 + 4p^3q^2 + p)b)$
- 164 $\psi((3p^5q^5 + 3p^5q^2 + 4p^3q^6 + 4p^3q^3 + pq)a + (8p^6q^4 + p^6q + 3p^4q^5 + 3p^4q^2 + 6p^2q^6 + 6p^2q^3 + p^2 + q^4)b)$
- 165 $\psi((8p^6q^5 + p^6q^2 + 3p^4q^6 + 3p^4q^3 + 6p^2q^4 + 7p^2q + q^5)a + (6p^5q^5 + 6p^5q^2 + p^3q^6 + p^3q^3 + p^3 + 2pq)b)$
- 166 $\psi((6p^5q^6 + 6p^5q^3 + p^3q^4 + 2p^3q + 2pq^2)a + (5p^6q^5 + 7p^6q^2 + 4p^4q^6 + 4p^4q^3 + p^4 + 6p^2q^4 + q^5)b)$
- 167 $\psi((5p^6q^6 + 7p^6q^3 + 4p^4q^4 + 5p^4q + 6p^2q^5 + q^6)a + (p^5q^6 + p^5q^3 + p^5 + 7p^3q^4 + 2p^3q + 6pq^5)b)$
- 168 $\psi((p^5q^4 + 2p^5q + 7p^3q^5 + 2p^3q^2 + 6pq^6)a + (6p^6q^6 + 8p^6q^3 + p^6 + 2p^4q^4 + 7p^4q + 3p^2q^5 + q^6)b)$
- 169 $\psi((8p^6q^4 + 7p^6q + 2p^4q^5 + 7p^4q^2 + 3p^2q^6 + q)a + (3p^5q^4 + p^3q^5 + 2p^3q^2 + 4pq^6 + 8pq^3 + p)b)$
- 170 $\psi((3p^5q^5 + p^3q^6 + 2p^3q^3 + 8pq^4 + 5pq)a + (2p^6q^4 + 7p^6q + 3p^4q^5 + 7p^2q^6 + 8p^2q^3 + p^2 + q)b)$
- 171 $\psi((2p^6q^5 + 7p^6q^2 + 3p^4q^6 + 8p^2q^4 + 8p^2q + q^2)a + (6p^5q^5 + 8p^3q^6 + p^3q^3 + p^3 + q^4 + 4pq)b)$
- 172 $\psi((6p^5q^6 + p^3q^4 + pq^5 + 4pq^2)a + (8p^6q^5 + 7p^6q^2 + 2p^4q^6 + p^4q^3 + p^4 + 3p^2q + q^2)b)$
- 173 $\psi((8p^6q^6 + 7p^6q^3 + p^4q^4 + 3p^4q + 3p^2q^2 + q^3)a + (8p^5q^6 + p^5q^3 + p^5 + p^3q^4 + 3p^3q + 3pq^2)b)$
- 174 $\psi((p^5q^4 + p^3q^5 + 3p^3q^2 + 3pq^3)a + (7p^6q^6 + 8p^6q^3 + p^6 + 2p^4q^4 + 6p^4q + 6p^2q^2q^3)b)$
- 175 $\psi((8p^6q^4 + 8p^6q + 2p^4q^5 + 6p^4q^2 + 6p^2q^3 + q^4)a + (3p^5q^4 + 6p^5q + p^3q^5 + 7pq^6 + 3pq^3 + p)b)$
- 176 $\psi((3p^5q^5 + 6p^5q^2 + p^3q^6 + 3pq^4 + 8pq)a + (2p^6q^4 + 5p^6q + 3p^4q^5 + 6p^4q^2 + 7p^2q^6 + p^2 + q^4)b)$
- 177 $\psi((2p^6q^5 + 5p^6q^2 + 3p^4q^6 + 6p^4q^3 + 8p^2q + q^5)a + (6p^5q^5 + 3p^5q^2 + 8p^3q^6 + p^3 + 6pq^4 + 4pq)b)$
- 178 $\psi((6p^5q^6 + 3p^5q^3 + 6pq^5 + 4pq^2)a + (8p^6q^5 + 8p^6q^2 + 2p^4q^6 + 6p^4q^3 + p^4 + 6p^2q^4 + 3p^2q + q^5)b)$

- 179 $\psi((8p^6q^6 + 8p^6q^3 + 6p^4q^4 + 3p^4q + 6p^2q^5 + 3p^2q^2 + q^6)a + (8p^5q^6 + p^5 + 6p^3q^4 + 3p^3q + 6pq^5 + 3pq^2)b)$
- 180 $\psi((6p^3q^5 + 3p^3q^2 + 6pq^6 + 3pq^3)a + (7p^6q^6 + 8p^6q^3 + p^6 + 3p^4q^4 + 6p^4q + 3p^2q^5 + 6p^2q^2 + q^6)b)$
- 181 $\psi((8p^6q^4 + 8p^6q + 3p^4q^5 + 6p^4q^2 + 3p^2q^6 + 6p^2q^3 + q)a + (3p^5q^4 + 6p^5q + 5pq^6 + 2pq^3 + p)b)$
- 182 $\psi((3p^5q^5 + 6p^5q^2 + 2pq^4 + 6pq)a + (2p^6q^4 + 5p^6q + 3p^4q^5 + 6p^4q^2 + 8p^2q^6 + 8p^2q^3 + p^2 + q)b)$
- 183 $\psi((2p^6q^5 + 5p^6q^2 + 3p^4q^6 + 6p^4q^3 + 8p^2q^4 + q^2)a + (6p^5q^5 + 3p^5q^2 + 8p^3q^6 + 8p^3q^3 + p^3 + 4pq^4 + 3pq)b)$
- 184 $\psi((6p^5q^6 + 3p^5q^3 + 8p^3q^4 + 4pq^5 + 3pq^2)a + (8p^6q^5 + 8p^6q^2 + 2p^4q^6 + 5p^4q^3 + p^4 + 3p^2q^4 + 3p^2q + q^2)b)$
- 185 $\psi((8p^6q^6 + 8p^6q^3 + 5p^4q^4 + 3p^4q + 3p^2q^5 + 3p^2q^2 + q^3)a + (8p^5q^6 + 8p^5q^3 + p^5 + 2p^3q^4 + 3p^3q + 3pq^5 + 3pq^2)b)$
- 186 $\psi((8p^5q^4 + 2p^3q^5 + 3p^3q^2 + 3pq^6 + 3pq^3)a + (7p^6q^6 + 7p^6q^3 + p^6 + 7p^4q^4 + 6p^4q + 6p^2q^5 + 6p^2q^2 + q^3)b)$
- 187 $\psi((7p^6q^4 + 8p^6q + 7p^4q^5 + 6p^4q^2 + 6p^2q^6 + 6p^2q^3 + q^4)a + (6p^5q^4 + 6p^5q + 8p^3q^5 + pq^6 + 2pq^3 + p)b)$
- 188 $\psi((6p^5q^5 + 6p^5q^2 + 8p^3q^6 + 2pq^4 + 2pq)a + (4p^6q^4 + 5p^6q + 6p^4q^5 + 6p^4q^2 + 7p^2q^6 + 8p^2q^3 + p^2 + q^4)b)$
- 189 $\psi((4p^6q^5 + 5p^6q^2 + 6p^4q^6 + 6p^4q^3 + 8p^2q^4 + 8p^2q + q^5)a + (3p^5q^5 + 3p^5q^2 + 6p^3q^6 + 8p^3q^3 + p^3 + 7pq^4 + 7pq)b)$
- 190 $\psi((3p^5q^6 + 3p^5q^3 + 8p^3q^4 + 7p^3q + 7pq^5 + 7pq^2)a + (7p^6q^5 + 8p^6q^2 + 3p^4q^6 + 5p^4q^3 + p^4 + 6p^2q^4 + 6p^2q + q^5)b)$
- 191 $\psi((7p^6q^6 + 8p^6q^3 + 5p^4q^4 + 4p^4q + 6p^2q^5 + 6p^2q^2 + q^6)a + (6p^5q^6 + 8p^5q^3 + p^5 + 5p^3q^4 + 4p^3q + 6pq^5 + 6pq^2)b)$
- 192 $\psi((8p^5q^4 + 7p^5q + 5p^3q^5 + 4p^3q^2 + 6pq^6 + 6pq^3)a + (4p^6q^6 + 7p^6q^3 + p^6 + p^4q^4 + 8p^4q + 3p^2q^5 + 3p^2q^2 + q^6)b)$
- 193 $\psi((7p^6q^4 + 5p^6q + p^4q^5 + 8p^4q^2 + 3p^2q^6 + 3p^2q^3 + q)a + (6p^5q + 8p^3q^5 + 7p^3q^2 + 2pq^6 + 4pq^3 + p)b)$
- 194 $\psi((6p^5q^2 + 8p^3q^6 + 7p^3q^3 + 4pq^4 + 3pq)a + (7p^6q^4 + 2p^6q + 6p^4q^2 + 5p^2q^6 + 7p^2q^3 + p^2 + q)b)$
- 195 $\psi((7p^6q^5 + 2p^6q^2 + 6p^4q^3 + 7p^2q^4 + 6p^2q + q^2)a + (3p^5q^2 + 4p^3q^6 + 5p^3q^3 + p^3 + 2pq^4 + 6pq)b)$
- 196 $\psi((3p^5q^3 + 5p^3q^4 + 5p^3q + 2pq^5 + 6pq^2)a + (7p^6q^5 + 5p^6q^2 + 4p^4q^6 + 2p^4q^3 + p^4 + 3p^2q + q^2)b)$
- 197 $\psi((7p^6q^6 + 5p^6q^3 + 2p^4q^4 + 5p^4q + 3p^2q^2 + q^3)a + (4p^5q^6 + 5p^5q^3 + p^5 + 5p^3q^4 + 8p^3q + 3pq^2)b)$
- 198 $\psi((5p^5q^4 + 5p^5q + 5p^3q^5 + 8p^3q^2 + 3pq^3)a + (2p^6q^6 + p^6q^3 + p^6 + 7p^4q^4 + 4p^4q + 6p^2q^2 + q^3)b)$
- 199 $\psi((p^6q^4 + 3p^6q + 7p^4q^5 + 4p^4q^2 + 6p^2q^3 + q^4)a + (3p^5q^4 + 5p^3q^5 + 5p^3q^2 + 2pq^6 + 5pq^3 + p)b)$
- 200 $\psi((3p^5q^5 + 5p^3q^6 + 5p^3q^3 + 5pq^4 + 3pq)a + (4p^6q^4 + 3p^6q + 3p^4q^5 + 2p^2q^6 + 2p^2q^3 + p^2 + q^4)b)$

- 201 $\psi((4p^6q^5 + 3p^6q^2 + 3p^4q^6 + 2p^2q^4 + 3p^2q + q^5)a + (6p^5q^5 + 7p^3q^6 + 7p^3q^3 + p^3 + pq^4 + 6pq)b)$
- 202 $\psi((6p^5q^6 + 7p^3q^4 + 8p^3q + pq^5 + 6pq^2)a + (p^6q^5 + 3p^6q^2 + p^4q^6 + 7p^4q^3 + p^4 + 3p^2q^4 + q^5)b)$
- 203 $\psi((p^6q^6 + 3p^6q^3 + 7p^4q^4 + 2p^4q + 3p^2q^5 + q^6)a + (7p^5q^6 + 7p^5q^3 + p^5 + p^3q^4 + 8p^3q + 3pq^5)b)$
- 204 $\psi((7p^5q^4 + 8p^5q + p^3q^5 + 8p^3q^2 + 3pq^6)a + (8p^6q^6 + p^6q^3 + p^6 + 8p^4q^4 + p^4q + 6p^2q^5 + q^6)b)$
- 205 $\psi((p^6q^4 + 8p^4q^5 + p^4q^2 + 6p^2q^6 + q)a + (6p^5q^4 + 7p^3q^5 + 8p^3q^2 + 3pq^6 + pq^3 + p)b)$
- 206 $\psi((6p^5q^5 + 7p^3q^6 + 8p^3q^3 + pq^4 + 4pq)a + (7p^6q^4 + 6p^4q^5 + p^2q^3 + p^2 + q)b)$
- 207 $\psi((7p^6q^5 + 6p^4q^6 + p^2q^4 + p^2q + q^2)a + (3p^5q^5 + 7p^3q^6 + p^3 + 8pq^4 + 5pq)b)$
- 208 $\psi((3p^5q^6 + 8p^3q + 8pq^5 + 5pq^2)a + (p^6q^5 + 4p^4q^6 + p^4 + 6p^2q + q^2)b)$
- 209 $\psi((p^6q^6 + 5p^4q + 6p^2q^2 + q^3)a + (7p^5q^6 + p^5 + 5p^3q + 6pq^2)b)$
- 210 $\psi((8p^5q + 5p^3q^2 + 6pq^3)a + (8p^6q^6 + p^6 + p^4q + 3p^2q^2 + q^3)b)$
- 211 $\psi((p^4q^2 + 3p^2q^3 + q^4)a + (8p^3q^2 + 8pq^6 + 7pq^3 + p)b)$
- 212 $\psi((8p^3q^3 + 7pq^4)a + (8p^2q^6 + p^2q^3 + p^2 + q^4)b)$
- 213 $\psi((p^2q^4 + q^5)a + (8p^3q^6 + p^3 + 8pq^4)b)$
- 214 $\psi(8pq^5a + (8p^4q^6 + p^4 + q^5)b)$
- 215 $\psi(q^6a + (8p^5q^6 + p^5)b)$
- 216 $\psi((8p^6q^6 + p^6 + q^6)b)$

The proof of this Theorem can be realized by induction, but in the present case the calculation of the coefficients of the base members is done by computer.

In conclusion we will mention that the extended Tribonacci sequence

$$T(0) = a, T(1) = b, T(2) = c,$$

$$T(n+3) = pT(n+2) + qT(n+1) + rT(n), \text{ for every natural number } n \geq 0,$$

where a, b, c, p, q, r are fixed natural numbers, has a base with respect to ψ -function with length 2808.

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