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Some Properties of Generalized Third Order Pell Numbers

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Abstract

This paper considers some properties of the third order recursive sequence defined by the linear recurrence relation

$$w_{m,n} = 2^m \ w_{m,n-2} + w_{m,n-3} \ , n \geq 3, \ m = 0,1,2,,$$

with appropriate initial conditions. The present work follows on from the case *m* = 0 (Shannon *et al*). Relationships with the well-known sequences of Fibonacci, Lucas and Pell are developed. The motivation for the study was to find analogous results to some of the second order classic identities such as, for example, Simson's identity and Horadam's Fibonacci number triples.

Keywords: Fibonacci, Lucas, Pell numbers, Simson, convolutions, generating functions.

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1. Introduction

In previous papers, we examined, *inter alia* some properties of the third order recursive sequence $\{r_n\}$ [2,8,9], defined by the linear recurrence relation

$$r_n = r_{n-2} + r_{n-3} \ , n \geq 3 \, , \tag{1.1}$$

with initial conditions $r_0 = 1, r_1 = 0, r_2 = 1$, and the second order recursive sequence $\{t_{m,n}\}$ [9], defined by the linear recurrence relation

$$t_{m,n} = 2^m t_{m,n-1} + t_{m,n-2}, \ n \geq 2 \, , \tag{1.2}$$

with initial conditions $t_{m,0} = 0, t_{m,1} = 1$. (In both cases we can allow for $n < 0$.) Furthermore, if we define $\{S_n\}$ to satisfy (1.1) with initial conditions $S_1 = 0, S_2 = 2, S_3 = 3$, we find that

$$S_n = a^n + b^n + \bar{b}^n \tag{1.3}$$

where $a, b, \bar{b}$ are the roots of the associated characteristic equation $x^3 = x + 1$ by analogy with the "Binet" form of the general term of the Lucas sequence, namely

$$L_n = \alpha^n + \beta^n \tag{1.4}$$

where α^n, β^n are the roots of the associated characteristic equation $x^2 = x + 1$.

In the same vein, we have

$$\{t_{0,n}\} \equiv \{F_n\}, \tag{1.5}$$

and

$$\{t_{1,n}\} \equiv \{P_n\}, \tag{1.6}$$

the sequence of Pell numbers [9].

2. Motivation

The motivation for this study arose from the neat analogies with the Fibonacci and Lucas number properties displayed by $\{r_n\}$. Some examples of these include

$$\begin{bmatrix} r_{-2} & r_0 & r_{-1} \\ r_{-1} & r_1 & r_0 \\ r_0 & r_2 & r_1 \end{bmatrix}^{n+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{n+1} = \begin{bmatrix} r_{n-2} & r_n & r_{n-1} \\ r_{n-1} & r_{n+1} & r_n \\ r_n & r_{n+2} & r_{n+1} \end{bmatrix}, n \geq 0 \tag{2.1}$$

in comparison with the well-known Fibonacci identity

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n+1} = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}, n \geq 0. \tag{2.2}$$

Similarly,

$$r_{n+j} = r_n r_j + r_{n-1} r_{j-1} + r_{n-1} r_{j-2} + r_{n-2} r_{j-1}, \tag{2.3}$$

by analogy with

$$f_{n+j} = f_n f_j + f_{n-1} f_{j-1}, \tag{2.4}$$

in which $f_n = F_{n+1}$ in the usual Fibonacci notation. Note that f_n count *n*-board tilings with squares and dominoes, and r_n count tilings with dominoes and triminoes [1].

It is proposed here to combine aspects of both the types of sequences in (1.1) and (1.2) by considering the third order recursive sequence defined by the linear recurrence relation

$$w_{m,n} = 2^m w_{m,n-2} + w_{m,n-3}, n \geq 3, \tag{2.5}$$

with initial conditions $w_{m,n}=1, w_{m,1}=m, w_{m,2}=2^m$. It can be readily confirmed then that

$$\{w_{0,n}\}\equiv\{r_n\},$$

and

$$\{w_{1,n}\}\equiv\{F_n\}.$$

The latter follows because

$$\begin{aligned} F_n &= F_{n-1}+F_{n-2} \\ &= \left(F_{n-2}+F_{n-3}\right)+F_{n-2} \\ &= 2F_{n-2}+F_{n-3}, \end{aligned}$$

3. The Case *m* = 1

Because the Fibonacci case is so well known we shall consider in this section the third order Pell-Fibonacci numbers by means of the third order recurrence relation

$$u_{1,n}=2u_{1,n-2}+u_{1,n-3},\ n\geq 3 \tag{3.1}$$

with initial conditions $u_{1,0}=u_{1,1}=u_{1,2}=1$, with the first few elements in Table 1.

N	0	1	2	3	4	5	6	7	8	9	10	11
$u_{1,n}$	1	1	1	3	3	7	9	17	25	43	67	111

Table 1: First few values of $\{u_{1,n}\}$

Consider now the associated characteristic equation:

$$f(x)\equiv x^3-2x-1=0 \tag{3.2}$$

which factors into

$$f(x)=(x+1)(x^2-x-1)$$

Consequently,

$$f(x)=(x-\alpha)(x-\beta)(x+1).$$

It is of interest to note here that Hall [3] formed a somewhat similar third order sequence with auxiliary equation roots $\alpha_1^2, \alpha_2^2, \alpha_1\beta_2$ from a second order sequence with auxiliary equation roots α_1 and α_2.

From the initial conditions, we can then get

$$u_{1,n+1}=\frac{2}{\alpha-\beta}\left(\alpha^n-\beta^n\right)+(-1)^n \tag{3.3}$$

or

$$u_{1,n}=2F_{n-1}-(-1)^n. \tag{3.4}$$

Moreover, if we also define $\{v_{1,\,n}\}$ to satisfy (1.3) with initial conditions $v_{1,\,0}=3,\ v_{1,\,1}=0,\ v_{1,\,2}=4$, then

$$v_{1,\,n}=\alpha^n+\beta^n+(-1)^n \tag{3.5}$$

$$=L_n+(-1)^n. \tag{3.6}$$

From (3.4) and (3.6) we obtain by analogy with

$$F_n\,L_n=F_{2n},$$

that

$$\begin{aligned}u_{1,\,n+1}\,v_{1,\,n}&=\left(2F_n+(-1)^n\right)\left(L_n+(-1)^n\right)\\&=2F_nL_n+1+(-1)^n\left(2F_n+L_n\right)\\&=2F_{2n}+1+(-1)^nF_{n+3},\end{aligned}$$

that is,

$$u_{1,\,n+1}\,v_{1,\,n}=u_{1,\,2\,n+1}+(-1)^nF_{n+3}. \tag{3.7}$$

For example, when n = 2, the left hand side of (3.7) is 3x4 and the right hand side is 7+5.

Another analogy is with Simson's identity as in (2.2), for which we have

$$\begin{vmatrix}u_{1,n-2}&u_{1,n}&u_{1,n-1}\\u_{1,n-1}&u_{1,n+1}&u_{1,n}\\u_{1,n}&u_{1,n+2}&u_{1,n+1}\end{vmatrix}=4 \tag{3.8}$$

Proof: We use induction on n and

$$\begin{bmatrix}0&1&0\\0&0&1\\1&2&0\end{bmatrix}\begin{bmatrix}u_{1,n-3}&u_{1,n-1}&u_{1,n-2}\\u_{1,n-2}&u_{1,n}&u_{1,n-1}\\u_{1,n-1}&u_{1,n+1}&u_{1,n}\end{bmatrix}=\begin{bmatrix}u_{1,n-2}&u_{1,n}&u_{1,n-1}\\u_{1,n-1}&u_{1,n+1}&u_{1,n}\\u_{1,n}&u_{1,n+2}&u_{1,n+1}\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&2&0\end{bmatrix}^n\begin{bmatrix}1&1&1\\1&3&1\\1&3&3\end{bmatrix}$$

Taking determinants we get the result.

4. Associated Polynomials

The $\{u_{1,\,n}\}$ can also be defined in terms of the ordinary generating function

$$\begin{aligned}u\,(y)&=\sum_{n=0}^{\infty}u_{1,\,n}\,y^n\\&=\left(1+y-y^2\right)\left(1-2y^2-y^3\right)^{-1}\quad.\\&=1+y+y^2+3y^3+3y^4+7y^5+\ldots\end{aligned} \tag{4.1}$$

This suggests a consideration of polynomials associated with $\{w_{m,n}\}$. We define associated polynomials

$$w_{m,n}(x) = 2^m w_{m,n-2}(x) + w_{m,n-3}(x), \; n \geq 3, \tag{4.2}$$

with $w_{m,0}(x) = 1, w_{m,1}(x) = mx, w_{m,2}(x) = 2^m x^2$. For reasons indicated above we shall consider $\{u_{1,n}(x)\}$ since variations of the Fibonacci polynomials are well known [7]. Consequently, we define polynomials $\{u_{1,n}(x)\}$:

$$u_{1,n}(x) = 2x^2 u_{1,n-2}(x) + u_{1,n-3}(x), \; n \geq 3, \tag{4.3}$$

with $u_{1,0}(x) = 1, u_{1,1}(x) = x, u_2(x) = x^2$. Then

$$\begin{aligned}
u_{1,3}(x) &= 2x^3 + 1 \\
u_{1,4}(x) &= 2x^4 + x \\
u_{1,5}(x) &= 4x^5 + 3x^2 \\
u_{1,6}(x) &= 4x^6 + 4x^3 + 1,
\end{aligned}$$

so that, in general,

$$u_{1,n}(x) = \sum_{i=0}^{\left\lfloor \frac{n+3}{3} \right\rfloor} a_{n,i} x^{n-3i} \tag{4.4}$$

in which $a_{n,i}$ satisfies the partial recurrence relation

$$a_{n,i} = 2a_{n-2,i} + a_{n-3,i-1}, 1 \leq i \leq \left\lfloor \frac{n+3}{3} \right\rfloor$$

with boundary conditions

$$a_{n,0} = 2^{\left\lfloor \frac{n-1}{2} \right\rfloor}, \; a_{n,i} = 0, \; i > \left\lfloor \tfrac{n+3}{3} \right\rfloor.$$

It is straightforward to establish that the ordinary generating function for $u_{1,n}(x)$ is

$$\begin{aligned}
U(x, y) &= \sum_{n=0}^{\infty} u_{1,n}(x) y^n \\
&= \left(1 + xy - xy^2\right)\left(1 - 2x^2 y^2 - y^3\right)^{-1} \\
&= 1 + xy + x^2 y^2 + \left(2x^3 + 1\right)y^3 + \left(2x^4 + x\right)y^4 + \ldots
\end{aligned} \tag{4.5}$$

and, from (4.1):

$$U(1, y) = U(y).$$

Riordan [6] utilized convolutions in his study of generating functions of powers of Fibonacci numbers, and Hoggatt and Bicknell-Johnson developed properties of Fibonacci convolution sequences [4]. More recently, Terrana and Chen [15] developed an alternative approach to that of Riordan. We now define *k*th order convolution polynomials, $\{U(x, y; k)\}$ by means of the generating function

$$\{U(x, y; k)\} = \sum_{n=0}^{\infty} u_{1,n}^{(k)}(x) y^n \tag{4.6}$$

$$= \left[(1 + xy - x^2 y^2)(1 - 2x^2 y^2 - y^3)^{-1}\right]^{k+1} \tag{4.7}$$

so that

$$U(x, y; 1) = U(x, y)$$

and

$$U(1, y; 1) = U(y).$$

From these we can obtain the first convolution numbers

$$U_{1,n}^{(1)} = \{1, 2, 3, 0, 11, 30, \ldots\} \tag{4.8}$$

and the interested reader might like to develop some properties for them such as finding an expression for

$$U_k(x) = \sum_{n=0}^{\infty} u_{1,n}^{k} x^n . \tag{4.9}$$

5. The Case *m* = 2

Here we outline some aspects of the case *m* = 2:

$$w_{2,n} = 4w_{2,n-2} + w_{2,n-3}, \quad n \geq 3, \tag{5.1}$$

with $w_{2,0} = 1$, $w_{2,1} = 2$ and $w_{2,2} = 2^2$. The first few terms are displayed in Table 2.

n	0	1	2	3	4	5	6	7	8	9
$w_{0,n}$	1	0	1	1	1	2	2	3	4	5
$w_{2,n}$	1	2	4	9	18	40	81	178	364	793

Table 2: The first ten terms of $\{w_{m,n}\}$, $m = 0, 2$.

The associated polynomials are then

$$w_{2,0}(x) = 1 \qquad = 1 \qquad = w_{0,2}(2x)$$

$$w_{2,1}(x) = 2x \qquad = (2x) \qquad = w_{0,3}(2x)$$

$$w_{2,2}(x) = 4x^2 \qquad = (2x)^2 \qquad = w_{0,4}(2x)$$

$$
\begin{aligned}
w_{2,3}(x) &= 8x^3+1 && =(2x)^3+1 && = w_{0,5}(2x)\\
w_{2,4}(x) &= 16x^4+2x && =(2x)^4+(2x) && = w_{0,6}(2x)\\
w_{2,5}(x) &= 32x^5+8x^2 && =(2x)^5+2(2x)^2 && = w_{0,7}(2x)\\
w_{2,6}(x) &= 64x^6+16x^3+1 && =(2x)^6+2(2x)^3+1 && = w_{0,8}(2x)\\
w_{2,7}(x) &= 128x^7+48x^4+2x && =(2x)^7+3(2x)^4+(2x) && = w_{0,9}(2x)\\
w_{2,8}(x) &= 256x^8+96x^5+12x^2 && =(2x)^8+3(2x)^5+3(2x)^2 && = w_{0,10}(2x)
\end{aligned}
$$

so that we can see

$$w_{2,n}(x) = w_{0,n+2}(2x)$$

which is analogous to the relation between the ordinary Fibonacci polynomials, $f_n(x)$, and the ordinary Pell polynomials, $p_n(x)$, namely,

$$p_n(x) = f_n(2x), \tag{5.2}$$

as in

$$p_6(x) = 32x^5 + 32x^3 + 6x = (2x)^5 + 4(2x)^3 + 3(2x) = f_6(2x).$$

Also

$$w_{0,n}(1) = w_{0,n} \text{ and } w_{2,n}(1) = w_{2,n}$$

by analogy with

$$f_n(1) = f_n$$

for the ordinary Fibonacci polynomials and numbers.

Somewhat like the $\{u_{1,n}(x)\}$ (in (4.4)), the $\{w_{0,n}(x)\}$satisfy

$$w_{0,n}(x) = \sum_{i=0}^{\left\lfloor \frac{n+3}{3} \right\rfloor} b_{n,i}\, x^{n-3i} \tag{5.3}$$

in which $b_{n,i}$ satisfies the partial recurrence relation

$$b_{n,i} = b_{n-2,i-1} + b_{n-1,i}, \; 1 \le i \le \left\lfloor \frac{n+3}{3} \right\rfloor$$

with boundary conditions

$$b_{n,0} = 1,\; b_{n,i} = 0,\; i > \left\lfloor \tfrac{n+3}{3} \right\rfloor.$$

We can also produce a Pythagorean triple analogous to Horadam's [5], which can be written as

$$\left(f_n(f_n + 2f_{n+1})\right)^2 + \left(2f_{n+1}f_{n+2}\right)^2 = \left(f_{2,n}^2 + 2f_{n+1}f_{n+2}\right)^2 \tag{5.4}$$

The corresponding result is

$$\left(\left(w_{m,n}\right)\left(w_{m,n}+2^{m+1}w_{m,n+1}\right)\right)^2+\left(2^{m+1}w_{m,n+1}w_{m,n+3}\right)^2 \quad =\left(w_{m,n}^2+2^{m+1}w_{m,n+1}w_{m,n+3}\right)^2 \tag{5.5}$$

Proof: From the recurrence relation (2.5) we have

$$\begin{aligned} w_{m,n}^2 &= \left(w_{m,n+3}-2^m u_{m,n+1}\right)^2 \\ &= w_{m,n+3}^2+2^{2m}w_{m,n+1}^2-2^{m+1}w_{m,n+1}w_{m,n+3}. \end{aligned}$$

Then

$$\begin{aligned} \left(w_{m,n}^2+2^{m+1}w_{m,n+1}w_{m,n+3}\right)^2 &= \left(w_{m,n+3}^2+2^{2m}w_{m,n+1}^2\right)^2 \\ &= w_{m,n+3}^4+2^{4m}w_{m,n+1}^4+2^{2m+1}w_{m,n+1}^2w_{m,n+3}^2 \\ &= \left(w_{m,n+3}^2-2^{2m}w_{m,n+1}^2\right)^2+2^{2m+2}w_{m,n+1}^2w_{m,n+3}^2 \\ &= \left(\left(w_{m,n+3}-2^m w_{m,n+1}\right)\left(w_{m,n+3}+2^m w_{m,n+1}\right)\right)^2+\left(2^{m+1}w_{m,n+1}w_{m,n+3}\right)^2 \\ &= \left(\left(w_{m,n}\right)\left(w_{m,n+3}+2^m w_{m,n+1}\right)\right)^2+\left(2^{m+1}w_{m,n+1}w_{m,n+3}\right)^2 \\ &= \left(\left(w_{m,n}\right)\left(w_{m,n}+2^{m+1}w_{m,n+1}\right)\right)^2+\left(2^{m+1}w_{m,n+1}w_{m,n+3}\right)^2. \end{aligned}$$

For instance, when $n = 0$ and $m = 2$ in (5.5), we have

$$\begin{aligned} \text{right hand side} &= \left(w_{2,0}(8w_{2,1}+w_{2,0}\right)^2+\left(8w_{2,1}w_{2,3}\right)^2 \\ &= 289+20{,}736 \\ &= 21{,}025; \\ \text{left hand side} &= \left(w_{2,0}^2+8w_{2,1}w_{2,3}\right)^2 \\ &= \left(1+8\times 2\times 9\right)^2 \\ &= 145^2 = 21{,}025. \end{aligned}$$

One could then investigate further in the manner of Horadam whether all Pythagorean triples can be represented by (5.5).

6. Concluding Comments

Other further research could involve the "golden ratio" associated with the Fibonacci sequence (m = 1) in the second column of Table 3 and the "plastic number" associated with the Padovan sequence (m = 0) in the fourth column of the table. This table also shows the first value of n for which the ratio stabilises at the stated level of accuracy. These ratios could be used to investigate "spirals" and to establish Binet-type general terms for the 3rd order sequences [13], as well as to reduce them to 2nd order sequences [10] and to find other properties such as those related to generalized continued fraction algorithms [9].

m	$t_{m,n+1} / t_{m,n}$	n	$u_{m,n+1} / u_{m,n}$	n
0	1.618	10	1.325	20
1	2.414	6	1.638	10
2	4.236	6	2.115	65
3	8.123	4	2.889	199

Table 3: Limiting ratios of adjacent terms (to 3 decimal places)

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