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Cycles of Binomial Coefficients

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Abstract: This paper considers some properties of rising and falling factorials by analogy with some classic results in number theory for cycles of binomial coefficients.

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1. Introduction

In numerous papers, Carlitz considered aspects of product cycles of binomial coefficients; for a representative sample of the range and style of these see [2,3,5,7,8,9,10]. In this paper, it is proposed to develop some analogies of there cycles for binomial coefficients built around rising and falling factorials.

2. Rising and Falling Factorials

In a previous paper [13], falling and rising factorials were utilized in the following forms. The falling factorial, an r-permutation of n distinct objects, is given by

$$\begin{aligned} n^{\underline{r}} &= P(n,r) \\ &= n(n-1)\ldots(n-r+1) \end{aligned} \quad (1.1)$$

and is such that

$$\begin{aligned} \nabla P(n,r) &= P(n,r) - P(n-1,r) \\ &= rP(n-1,r-1). \end{aligned} \quad (1.2)$$

Similarly, we showed for the rising factorial of n

$$n^{\overline{r}} = n(n+1)\ldots(n+r-1) \quad (1.3)$$

that

$$\begin{aligned} \nabla n^{\overline{r}} &= n^{\overline{r}} - (n-1)^{\overline{r}} \\ &= rn^{\overline{r-1}} \end{aligned} \quad (1.4)$$

This is a recurrence relation for $n^{\bar{r}}$, which is an r permutation of $n+r-1$ objects, and which is related to the Stirling numbers. Corresponding binomial coefficients were also considered, namely.

$$\binom{n}{r} = \frac{n(n-1)...(n-r+1)}{r(r-1)...(r-r+1)} = \frac{n^{\underline{r}}}{r^{\underline{r}}} \tag{1.5}$$

in which $n^{\underline{r}}$ is the falling r-factorial of n and

$$C(n,r) = \frac{n^{\bar{r}}}{r^{\bar{r}}} \tag{1.6}$$

In which $n^{\bar{r}}$ is the rising r-factorial of n. Thus

$$C(n,r) = \frac{n(n+1)...(n+r-1)}{r(r+1)...(r+r-1)}, \tag{1.7}$$

which is also suggested by the Gauss-Cayley form of the generalized binomial coefficient. Here it is proposed to illustrate and extend Carlitz' approach to cycles of binomial coefficients.

3. Rising Binomial Coefficients

Carlitz [7] uses the fact that

$$\sum_{j=0}^{\infty}\binom{i+j}{j}v^{j} = \frac{1}{(1-v)^{i+1}} \tag{3.1}$$

Ana analogue of this can be readily developed from Carlitz and Riordan [11]. Put $m = j$, $x = 1$, $n = i$, $y = v$ in Equation (7.4) of this last paper, and it becomes

$$\sum_{j=0}^{\infty}\begin{bmatrix} i+j \\ j \end{bmatrix} = \frac{1}{(v)_{i+1}} \tag{3.2}$$

in which we have Carlitz' q-series analogue of the binomial coefficient defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k(q)_{n-k}} \tag{3.3}$$

where

$$(q)_n = (1-q)(1-q^2)...(1-q^n). \tag{3.4}$$

This suggests that we define a rising factorial analogue of the binomial coefficient in a similar manner an then proceed to examine some of its properties. We define the rising binomial coefficient – type I:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_a = \frac{a^n}{a^k a^{n-k}} \tag{3.5}$$

We can also define a rising binomial coefficient – type II in a similar manner:

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_z = \frac{n^{\bar{z}}}{k^{\bar{z}}(n-k)^{\bar{z}}} \tag{3.6}$$

Since, when $a = 1$,

$$\left\{ \begin{matrix} i+j \\ j \end{matrix} \right\}_a = \binom{i+j}{j}.$$

from (3,1) we have that

$$\sum_{j=0}^{\infty} \frac{1^{\overline{i+j}}}{1^{\bar{i}} 1^{\bar{j}}} v^j = \frac{1}{(1-v)^{i+1}}.$$

It can also be shown that

$$\sum_{j=0}^{\infty} \frac{(-1)^i}{\left(-(1)^{\bar{i}}\right)^{j+1}} v^{j-1} = \frac{1}{v^{\overline{i+1}}}. \tag{3.7}$$

Proof:

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{(-1)^i}{\left(-(1)^{\bar{i}}\right)^{j+1}} v^{j-1} &= v^{-1} \sum_{j=0}^{\infty} \frac{(-1)^{i+j+1}}{(i!)^{j+1}} v^j \\
&= \frac{1}{v} \sum_{j=0}^{\infty} (-1)^{i+j-1} v^j \prod_{r=0}^{i-1} \frac{1}{(i-r)^{j+1}} \\
&= \prod_{r=0}^{i-1} \frac{(-1)^{i-1}}{v(i-r)} \sum_{j=0}^{\infty} \left(\frac{v}{r-i} \right)^j \\
&= \prod_{r=0}^{i-1} \frac{(-1)^i}{v(i-r)} \left(1 - \frac{v}{r-i} \right)^{-1} \\
&= \prod_{r=0}^{i-1} (-1)^i v^{-1} (r-i-v)^{-1} \\
&= \prod_{r=0}^{i-1} v^{-1} (v+i-r)^{-1} \\
&= \prod_{r=0}^{i-1} (v+i-r)^{-1} \\
&= \frac{1}{v(v+1)\ldots(v+i)} \\
&= \frac{1}{v^{\overline{i+1}}}
\end{aligned}$$

4. Cycles of Length 2 of Mixed Coefficients

Carlitz' approach to cycles of binomial coefficients can also be illustrated by investigating a cycle of length two of mixed coefficients.

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\begin{bmatrix} i+j \\ j \end{bmatrix}\binom{j+k}{k}v^j w^k = \sum_{j=0}^{\infty}\begin{bmatrix} i+j \\ j \end{bmatrix}v^j(1-w)^{-j-1}$$
$$= \frac{1}{1-w}\sum_{j=0}^{\infty}\begin{bmatrix} i+j \\ j \end{bmatrix}\left(\frac{v}{1-w}\right)^j$$
$$= \left((1-w)\left(\frac{1-w}{v}\right)_{i+1}\right)^{-1}. \qquad (4.1)$$

These cannot be readily developed further because of the crucial step [12] in forming the cycle which follows. This is outlined and developed in Riordan and Carlitz [6] .

$$\left[(1-v-w)-(1-w)uw^{-1}\right]^{-1} = (1-v-w)^{-1}\left[1-\frac{u(1-w)}{v(1-v-w)}\right]^{-1}$$
$$= \sum_{i=0}^{\infty}u^i w^{-1}(1-w)^i(1-v-w)^{-i-1}$$
$$= \sum_{i,j,k=0}^{\infty}\binom{i+j}{j}\binom{j+k}{k}u^i v^j w^{k-i}.$$

The coefficient of w^0 is the generating function

$$\sum_{i,j=0}^{\infty}\binom{i+j}{j}\binom{j+i}{i}u^i v^j.$$

Also,

$$\left[1-v-w-(1-w)uw^{-1}\right]^{-1} = \left[1+u-v-(w+uw^{-1})\right]^{-1}$$
$$= \sum_{r=0}^{\infty}(w+uw^{-1})^r(1+u=v)^{-r-1}$$
$$= \sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}u^s w^{r-2s}(1+u-v)^{-r-1}$$

and the coefficients of w^0 in this is

$$\sum_{r=0}^{\infty}\binom{2r}{r}u^r(1+u-v)^{-2r-1} = \left[(1+u-v)^2-4u\right]^{\frac{1}{2}}.$$

Thus,

$$\sum_{i,j=0}^{\infty}\binom{i+j}{j}^2 u^i v^j = \left[(1+u-v)^2-4u\right]^{\frac{1}{2}} \tag{4.2}$$

$$= \left[(1-u-v)^2-4uv\right]^{\frac{1}{2}}$$

$$= q(uv^{-1},v)$$

where

$$q(x,y) = \sum_{n,m=0}^{\infty}\binom{n+m}{n}^2 x^n y^m$$

is an associated Legendre polynomial. It is related to the Legendre polynomial by means of

$$q_n(x) = \sum_{k=0}^{n}\binom{n}{k}^2 x^k$$

$$= (1-x)^n P_n\left[(1+x)(1-x)^{-1}\right]$$

$$\sum_{n=0}^{\infty} q_n(x) y^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2 x^k y^n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2 x^k y^k y^{n-k}$$

$$= \sum_{n,m=0}^{\infty}\binom{n+m}{n} x^n y^m$$

$$= q(x,y).$$

Thus

$$q(x,y) = \sum_{n=0}^{\infty}(1-x)^n p_n\left[(1+x)(1-x)^{-1}\right] \tag{4.2}$$

It is of interest to put $k = i$ and $w = u$ in and compare the results with, we get

$$\sum_{i,j=0}^{\infty}\begin{bmatrix} i+j \\ j \end{bmatrix}\binom{i+j}{j} u^i v^j = \left((1-u)\left(\frac{1-u}{v}\right)_{i+1}\right)^{-1}. \tag{4.3}$$

5. Conclusion

To complete this paper we note an identity of Carlitz:

$$\sum_{s=-m}^{m}(-1)^s\begin{bmatrix} m+n \\ m+s \end{bmatrix}\begin{bmatrix} n+p \\ n+s \end{bmatrix}\begin{bmatrix} p+m \\ p+s \end{bmatrix}q^{\frac{1}{2}s(3s+1)} = \frac{(\underline{q}_{m+n+p})!}{\underline{q}_m!\underline{q}_n!\underline{q}_p!}$$

For the interested reader this might suggest extending these identities to the equivalent multinomial coefficients [4].

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