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Note on Polynomials Taking Infinitely Many Primes as Their Values

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Abstract

In the paper the polynomials with integer coefficients are considered and a hypothetical sufficient condition for these polynomials to take infinitely many primes as their values is proposed. That provides an alternative equivalent variant of the famous Dirichlet's theorem for infinitely many primes in arithmetic progressions. Also an interesting analogy between the behaviour of polynomial's zeros and the integers for which the polynomial takes prime values is noted.

Let us consider an infinite arithmetic progression $a + k.d$, $k = 0, 1, 2, \dots,$ where $a \neq 0$ is integer and d is positive integer. The famous Dirichlet's theorem (see [1]) states that this progression contains infinitely many primes if and only if a and d are relative primes.

We note that the above sequence may be considered as a restriction of the linear function

$$f(x) = a + d.x \tag{1}$$

over all positive integers x.

It's easy to see that if:

$$f(x_1) = p_1,\ f(x_2) = p_2,$$

where p_i, $i = 1, 2$, are different primes and x_i, $i = 1, 2$, are positive integers, then $f(x)$ takes infinitely many different primes as its values over the set of all positive integers. Indeed, if we suppose that there exists a prime p such that p divides a and d simultaneously, then p is a divisor of p_1 and p is a divisor of p_2, which is impossible. So, under the above assumption we have that a and d are relative primes, hence we may apply Dirichlet's theorem to conclude that the linear function $f(x)$ from (1) takes infinitely many primes as its values over the set of all positive integers.

Thus, we proved the following assertion that gives us an unexpected equivalent form of Dirichlet's theorem.

Theorem 1. Let $a \neq 0$ be integer and d be positive integer. Then the linear function $f(x)$ from (1) takes infinitely many primes as its values over the set of all positive integers if and only if $f(x)$ takes two different primes as its values over the set of all positive integers.

Since the linear function $f(x)$ is polynomial of degree one with integer coefficients and positive leading coefficient, we decide completely (with the help of Theorem 1) the question when such polynomials take infinitely many primes as their values. So it is natural to ask the same question but for polynomials of higher degrees. It is clear that the formulation of Theorem 1 suggests the following general hypothetical

Theorem 2. Let $f(x)$ be a polynomial of degree $n \geq 1$, with integer coefficients and positive leading coefficient. Then $f(x)$ takes infinitely many primes as its values over the set of all positive integers if and only if, $f(x)$ takes $n + 1$ different primes as its values over the set of all positive integers.

It is known (see [2]) that if $f(x)$ is a polynomial with integer coefficients and with positive leading coefficient, then the necessary condition for it to take infinitely many prime numbers as its values over the set of all positive integers is the simultaneous fulfillment of the following two requirements:

1) $f(x)$ is irreducible over the set of integers.

2) For every positive integer $a > 1$ there exists an integer $x = x(a)$ such that $f(x(a))$ is not divisible by a.

The question whether the above requirements are also a sufficient condition remains unsolved. The final generalization in this direction is the

following result [3]:
Schinzel's Hypothesis. Let s be a positive integer, and $f_1(x), \ldots, f_s(x)$ are irreducible polynomials with integer coefficients and with positive leading coefficients. If there exists no integer $n > 1$ such that it divides $\prod_{k=1}^{s} f_k(x)$ for all integers x, then there should exist infinitely many positive integers x such that the numbers $f_1(x), \ldots, f_s(x)$ are simultaneously primes.

We must note that Theorem 2 gives a different, from the above mentioned, and simpler criterion whether a polynomial $f(x)$ of degree n with integer coefficients and with positive leading coefficient takes infinitely many prime numbers as its values over the the set of all positive integers.

As a corollary from the Fundamental Theorem of Algebra (see [4]) the following assertion may be formulated:
Theorem 3. If $f(x)$ is an arbitrary polynomial of degree n over the complex number field which becomes zero in $n+1$ different values of the argument x, then $f(x)$ becomes zero for infinitely many different values of the argument x.

Remark. Under the conditions of Theorem 3 the polynomial $f(x)$ is identically equal to 0.

If one compares Theorem 2 to Theorem 3 a certain remarkable similarity may be seen. Namely, the values in $n+1$ points determine the behaviour of the polynomial of degree n in both cases. In Theorem 2, the integers for which the polynomial takes prime numbers as its values act just like the polynomial zeros from Theorem 3.

References

[1] Dirichlet, P.G.L., R., Dedekind *Vorlesungen uber Zahlentheorie.* Chelsea, New York, 1968

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[3] Weisstein, Eric W. "Schinzel's Hypothesis." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/SchinzelsHypothesis.html

[4] Weisstein, Eric W. "Fundamental Theorem of Algebra." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/FundamentalTheoremofAlgebra.html