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Extension of an elementary equality

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Let $k \geq 1$ be a natural number and let

$$V_k = \{(i_1, i_2, ..., i_{3k}) : (i_1, i_2, ..., i_{3k}) \text{ is a permutation of } (1, 2, ..., 3k)\}.$$

Here we shall formulate and prove the following
Theorem. Let $u_1, u_2, ..., u_{3k}$ be complex numbers satisfying equality

$$\prod_{i=1}^{3k} u_i = 1. \tag{1}$$

Then

$$\prod_{(i_1,i_2,...,i_{3k})\in V_k} \{(\prod_{j=1}^{k} u_{i_j} - 1 + \frac{1}{\prod_{j=k+1}^{2k} u_{i_j}})$$
$$.(\prod_{j=k+1}^{2k} u_{i_j} - 1 + \frac{1}{\prod_{j=2k+1}^{3k} u_{i_j}}).(\prod_{j=2k+1}^{3k} u_{i_j} - 1 + \frac{1}{\prod_{j=1}^{k} u_{i_j}})\}$$
$$= \prod_{(i_1,i_2,...,i_{3k})\in V_k} \{(\prod_{j=1}^{k} u_{i_j} + 1 - \frac{1}{\prod_{j=k+1}^{2k} u_{i_j}})$$
$$.(\prod_{j=k+1}^{2k} u_{i_j} + 1 - \frac{1}{\prod_{j=2k+1}^{3k} u_{i_j}}).(\prod_{j=2k+1}^{3k} u_{i_j} + 1 - \frac{1}{\prod_{j=1}^{k} u_{i_j}})\}. \tag{2}$$

Proof. Let complex numbers $u_1, u_2, ..., u_{3k}0$ satisfying (1) be given and let $(i_1, i_2, ..., i_{3k}) \in V_k$ be fixed. Therefore, we can put

$$a = \prod_{j=1}^{k} u_{i_j},$$

$$b = \prod_{j=k+1}^{2k} u_{i_j},$$

$$c = \prod_{j=2k+1}^{3k} u_{i_j} = \frac{1}{\prod_{j=k+1}^{2k} u_{i_j}} = \frac{1}{ab}.$$

Let complex numbers x, y and z be determined in a way that

$$a = \frac{x}{y},\ b = \frac{y}{z},\ c = \frac{z}{x}.$$

Then we see that for fixed $(i_1, i_2, ..., i_{3k}) \in V_k$:

$$(\prod_{j=1}^{k} u_{i_j} - 1 + \frac{1}{\prod_{j=k+1}^{2k} u_{i_j}}).(\prod_{j=k+1}^{2k} u_{i_j} - 1 + \frac{1}{\prod_{j=2k+1}^{3k} u_{i_j}}).(\prod_{j=2k+1}^{3k} u_{i_j} - 1 + \frac{1}{\prod_{j=1}^{k} u_{i_j}})$$

$$= (a - 1 + \frac{1}{b}).(b - 1 + \frac{1}{c}).(c - 1 + \frac{1}{a})$$

$$= (\frac{x}{y} - 1 + \frac{z}{y}).(\frac{y}{z} - 1 + \frac{x}{z}).(\frac{z}{x} - 1 + \frac{y}{x})$$

$$= \frac{1}{xyz}.(x - y + z).(y - z + x).(z - x + y)$$

$$= \frac{1}{xyz}.(x + y - z).(y + z - x).(z + x - y)$$

$$= (\frac{x}{y} + 1 - \frac{z}{y}).(\frac{y}{z} + 1 - \frac{x}{z}).(\frac{z}{x} + 1 - \frac{y}{x})$$

$$= (a + 1 - \frac{1}{b}).(b + 1 - \frac{1}{c}).(c + 1 - \frac{1}{a})$$

$$= (\prod_{j=1}^{k} u_{i_j} + 1 - \frac{1}{\prod_{j=k+1}^{2k} u_{i_j}}).(\prod_{j=k+1}^{2k} u_{i_j} + 1 - \frac{1}{\prod_{j=2k+1}^{3k} u_{i_j}}).(\prod_{j=2k+1}^{3k} u_{i_j} + 1 - \frac{1}{\prod_{j=1}^{k} u_{i_j}})$$

with which (2) is proved.

In the particular case $k = 1$ we obtain that for three arbitrary complex numbers a, b and c for which $abc = 1$, the equality

$$(a - 1 + \frac{1}{b}).(b - 1 + \frac{1}{c}).(c - 1 + \frac{1}{a}) = (a + 1 - \frac{1}{b}).(b + 1 - \frac{1}{c}).(c + 1 - \frac{1}{a}) \qquad (3)$$

holds. It extends the equality introduced by the author in [1], where a, b and c are positive real numbers.

We must note that the opposite assertion *if (3) is valid for arbitrary positive real numbers a, b and c, then (1) is valid* does not hold. Because, for example, if

$$a = \frac{4}{3},\ b = \frac{1}{3},\ c = \frac{1}{3}$$

then

$$(a - 1 + \frac{1}{b}).(b - 1 + \frac{1}{c}).(c - 1 + \frac{1}{a}) = \frac{70}{108} = (a + 1 - \frac{1}{b}).(b + 1 - \frac{1}{c}).(c + 1 - \frac{1}{a}),$$

but

$$a.b.c = \frac{4}{27} \neq 1.$$

References

[1] Atanassov K., An elementary identity. *Notes on Number Theory and Discrete Mathematics,* Vol. 6 (2000), No. 3, 100.