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Some Relations Involving Bernoulli Numbers

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Abstract In the present note some identities involving Bernoulli numbers and binomial coefficients are considered.
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As it is known, Bernoulli numbers (see [1]) are denoted by $B_m, m = 0, 1...$ and are introduced by:

$$B_0 = 1;$$
$$\sum_{p=1}^{m} \binom{m+1}{p} B_p = 0, m \geq 1$$

In particular we have $B_0 = 1, B_1 = \frac{-1}{2}, B_2 = \frac{1}{6}, B_4 = \frac{-1}{30}, B_6 = \frac{1}{42}, B_8 = \frac{-1}{30}, B_{10} = \frac{5}{66}$, $B_{12} = \frac{-691}{2730}, B_{14} = \frac{7}{6}, B_{16} = \frac{-3617}{510}$, etc. Also :

$$B_{2t+1} = 0, t \geq 1$$

Let us define a bivariable polynomial $f_k(x, y)$ by:

$$f_k(x,y) := \sum_{t=1}^{p} (-1)^p \binom{p}{t} x^{k+1-t} y^t \qquad (1),$$

where $k \geq 1$ is an integer. Then $deg f_k(x, y) = k + 1$, where deg signifies the degree of the polynomial. From [2] the following results are seen:

Lemma. The polynomial $f_k(x, y)$ from (1) is a symmetric function with respect to x and y, i.e the equality:

$$f_k(x,y) = f_k(y,x) \qquad (2),$$

holds. Also $f_k(x, y)$ admits the representation:

$$f_k(x,y) = x^{k+1} + \sum_{p=1}^{k} \binom{k+1}{p} B_p (x-y)^p x^{k+1-p}$$

Theorem 1. Let $k \geq 1$ be an integer. Then for $\alpha = 1, 2..., k$ the identity:

$$\sum_{p=\alpha}^{k} \binom{p}{\alpha} B_p = (-1)^{k+1} \sum_{p=k+1-\alpha}^{k} \binom{k+1}{p} \binom{p}{k+1-\alpha} B_p \qquad (3),$$

holds.

Proof. From (1) the coefficient preceeding $x^{k+1-\alpha}y^{\alpha}$ is equal to

$$(-1)^{\alpha}\sum_{p=\alpha}^{k}\binom{k+1}{p}\binom{p}{\alpha}B_p,$$

and the coefficient preceeding $x^{\alpha}y^{k+1-\alpha}$ is equal to

$$(-1)^{k+1-\alpha}\sum_{p=k+1-\alpha}^{k}\binom{k+1}{p}\binom{p}{\alpha}B_p.$$

Hence (3) is true since (2) holds. Using that:

$$\binom{k+1}{p}\binom{p}{\alpha}=\binom{k+1}{\alpha}\binom{k+1-\alpha}{p-\alpha};$$

$$\binom{k+1}{p}\binom{p}{k+1-p}=\binom{k+1}{\alpha}\binom{\alpha}{k+1-p},$$

we obtain from (3):

$$\sum_{p=\alpha}^{k}\binom{k+1-\alpha}{p-\alpha}B_p=\sum_{p=k+1-\alpha}^{k}(-1)^{k+1}\binom{\alpha}{k+1-p}B_p \qquad (4),$$

which is valid for $\alpha=1,2,...,k$.

The Theorem is proved.

We can now formulate:

Theorem 2. For every two positive integers α and β the identity:

$$(-1)^{\beta}(\binom{\beta}{0}B_{\alpha}+...+\binom{\beta}{\beta-1}B_{\alpha+\beta-1})=(-1)^{\alpha}(\binom{\alpha}{0}B_{\beta}+...+\binom{\alpha}{\alpha-1}B_{\alpha+\beta-1}) \quad (5),$$

holds.

Proof.

Putting in (4) $k=\alpha+\beta-1$ we obtain the identity:

$$\binom{\beta}{0}B_{\alpha}+...+\binom{\beta}{\beta-1}B_{\alpha+\beta-1}=(-1)^{\alpha+\beta}(\binom{\alpha}{0}B_{\beta}+...+\binom{\alpha}{\alpha-1}B_{\alpha+\beta-1}) \qquad (6),$$

Then (6) yields immediately the equation (5) and also the identity:

$$(-1)^{\alpha}(\binom{\beta}{0}B_{\alpha}+...+\binom{\beta}{\beta-1}B_{\alpha+\beta-1})=(-1)^{\beta}(\binom{\alpha}{0}B_{\beta}+...+\binom{\alpha}{\alpha-1}B_{\alpha+\beta-1}) \quad (7),$$

The Theorem is proved.

Assuming that the symbol $\binom{0}{m}=0,$ for $m\geq 0$, and $\binom{0}{-1}=0$ then, (5),(6) and (7) remain valid for each of the cases $\alpha=0,\beta\neq 0$ and $\alpha\neq 0,\beta=0$, because of the relation defining the Bernoulli numbers for $m\geq 1$. Moreover, (5), (6) and (7) will remain valid for the case $\alpha=0,\beta=0$. But then these identities are trivial.

The above means that each one of (5),(6) and (7) is a generalization of the defining property for Bernoulli numbers

$$\sum_{p=0}^{m}\binom{m+1}{p}B_p=0.$$

References

[1] Borevich Z., I. Shafarevich , Number Theory, M., 1972, (in Russian)

[2] Vassilev P.,M. Vassilev-Missana , On the Sum of Equal Powers of the First n Terms of an Arithmetic Progression, Notes on Number Theory and Discrete Mathematics, Vol. 11, No. 3, 2005, 15-21.