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RISING FACTORIAL BERNOULLI POLYNOMIALS

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Abstract:
This paper considers some properties of rising binomial coefficients and two analogs of the Bernoulli polynomials which can be developed from them.

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1. Introduction

In a previous paper [7], falling and rising factorials were utilised in the following forms. The falling factorial, an r-permutation of n distinct objects, is given by

$$\begin{aligned} n^{\underline{r}} &= P(n, r) \\ &= n(n-1)\ldots(n-r+1) \end{aligned} \tag{1.1}$$

and is such that

$$\begin{aligned} \nabla P(n,r) &= P(n,r) - P(n-1,r) \\ &= rP(n-1,r-1), \end{aligned} \tag{1.2}$$

(see, for example Riordan [5]). Here we have followed Knuth's suggestion at the 1967 Conference on Combinatorial Mathematics and its Applications (Riordan [6]) for the rising and falling factorial coefficients. Similarly, we showed for the rising factorial of n

$$n^{\overline{r}} = n(n+1)\ldots(n+r-1) \tag{1.3}$$

that

$$\begin{aligned} \nabla n^{\overline{r}} &= n^{\overline{r}} - (n-1)^{\overline{r}} \\ &= rn^{\overline{r-1}}. \end{aligned} \tag{1.4}$$

This is a recurrence relation for $n^{\overline{r}}$, which is an r permutation of n + r - 1 objects, and which is related to the Stirling numbers. Corresponding binomial coefficients were also considered, namely,

$$\binom{n}{r} = \frac{n(n-1)...(n-r+1)}{r(r-1)...(r-r+1)} = \frac{n^{\underline{r}}}{r^{\underline{r}}} \tag{1.5}$$

in which $n^{\underline{r}}$ is the falling r-factorial of n and

$$C(n,r) = \frac{n^{\overline{r}}}{r^{\overline{r}}} \tag{1.6}$$

in which $n^{\overline{r}}$ is the rising r-factorial of n. Thus

$$C(n,r) = \frac{n(n+1)...(n+r-1)}{r(r+1)...(r+r-1)}, \tag{1.7}$$

which is also suggested by the Gauss-Cayley form of the generalized binomial coefficient. Here it is proposed to consider two rising factorial Bernoulli polynomials, $B(n,z,t)$ and $B'(n,r,z)$ which are defined in (2.1) and (3.1) below.

2. Rising Factorial Bernoulli Polynomial I: *B*(*n*,*z*,*t*)

We define a rising factorial Bernoulli polynomial I, $B(n,z,t)$, by

$$\frac{te(z,xt)}{e(z,t)-1} = \sum_{n=0}^{\infty} B(n,z,x)\frac{t^n}{z^{\overline{n}}} \tag{2.1}$$

in which $e(z,x)$ is an analog of the exponential function:

$$e(z,x) = \sum_{n=0}^{\infty} \frac{x^n}{z^{\overline{n}}} \tag{2.2}$$

We can see the analogies with the ordinary exponential function and Bernoulli polynomial in that $e(1,x) = e^x$ and $B(n,1,x) = B_n(x)$. The main result is that

$$x^n = \sum_{k=0}^{n} \frac{1}{k+z} \begin{Bmatrix} n \\ k \end{Bmatrix}_z B(n-k,z,x) \tag{2.3}$$

where we have used the rising binomial coefficient – type I:

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_a = \frac{a^{\overline{n}}}{a^{\overline{k}} a^{\overline{n-k}}} \tag{2.4}$$

and so (2.3) is the analog of Carlitz [2]:

$$x^n = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} B_{n-k}(x) \tag{2.5}$$

for the ordinary Bernoulli numbers. The proof of (2.3) now follows:

$$
\begin{aligned}
\frac{e(z,t)-1}{t} &= \frac{1}{t}\sum_{n=1}^{\infty}\frac{t^n}{z^{\overline{n}}} \\
&= \sum_{n=1}^{\infty}\frac{t^{n-1}}{z^{\overline{n}}} \\
&= \sum_{n=0}^{\infty}\frac{t^n}{z^{\overline{n}}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
e(z,xt) &= \sum_{n=0}^{\infty}\frac{x^n t^n}{z^{\overline{n}}} \\
&= \sum_{m=0}^{\infty}\frac{t^m}{z^{\overline{m+1}}}\sum_{n=0}^{\infty}\frac{B(n,z,x)t^n}{z^{\overline{n}}} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{B(n-k,z,x)t^n}{(k+z)z^{\overline{k}}z^{\overline{n-k}}} \\
&= \sum_{n=0}^{\infty}\frac{t^n}{z^{\overline{n}}}\sum_{k=0}^{n}\frac{1}{k+z}\left\{\begin{matrix} n \\ k \end{matrix}\right\}_z B(n-k,z,x)
\end{aligned}
$$

and the result follows on equating coefficients of x.

3. Rising Factorial Bernoulli Polynomial II: *B'*(*N*,*R*,Z)

From the definition of $e(n,z,x)$ we get

$$
\begin{aligned}
\frac{e(n,z,t)-\frac{1}{z!}}{t} &= \frac{1}{t}\left\{\sum_{n=0}^{\infty}\frac{t^n}{(n+1)^{\overline{z}}}-\frac{1}{z!}\right\} \\
&= \sum_{n=1}^{\infty}\frac{t^{n-1}}{(n+1)^{\overline{z}}} \\
&= \sum_{n=0}^{\infty}\frac{t^n}{(n+2)^{\overline{z}}}. \qquad (3.1)
\end{aligned}
$$

Thus,

$$e(n,z,xt) = \sum \frac{t^n x^n}{(n+1)^{\bar{z}}}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{k(k+1)B'(n-k,z,x)}{(k+z)(k+z+1)k^{\bar{z}}(n-k+1)^{\bar{z}}}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{(n+1)^{\bar{z}}} \sum_{k=0}^{n} \frac{k(k+1)}{(k+z)(k+z+1)} \left\langle {n+1 \atop k} \right\rangle_z B'(n-k,z,x),$$

in which

$$\left\langle {n \atop k} \right\rangle_z = \frac{n^{\bar{z}}}{k^{\bar{z}}(n-k)^{\bar{z}}}$$

is the rising binomial coefficient – type II. We then get

$$x^n = \sum_{k=0}^{n} \frac{k(k+1)}{(k+z)(k+z+1)} \left\langle {n+1 \atop k} \right\rangle_z B'(n-k,z,x), \tag{3.2}$$

as another analog of the result in (2.5) for the ordinary Bernoulli numbers,

4. Concluding Comments

When a is replaced by q in the rising binomial coefficient defined in (2.4) we have Carlitz' definition of the q-series analog of the binomial coefficient [1].

We also observe that Sylvester [8] defined the nth Fermatian of index q by

$$\underline{q}_n = \begin{cases} -q^n \underline{q}_{-n} & (n<0) \\ 1 & (n=0) \\ 1+q+q^2+\ldots+q^{n-1} & (n>0) \end{cases} \tag{4.1}$$

The contrasting of the Fermatians and the q-series is not as bizarre as it might seem at first sight. Thus, as an example, the favourite Ramanujan identity of Dyson [3]

$$\sum_{n=0}^{\infty} x^{n^2+n} \frac{(1+x+x^2)(1+x^2+x^4)\ldots(1+x^n+x^{2n})}{(1-x)(1-x^2)\ldots(1-x^{2n+1})} = \prod_{n=1}^{\infty} \frac{(1-x^{2n})}{(1-x^n)}$$

can also be written as

$$\sum_{n=0}^{\infty} x^{n^2+n} \prod_{m=1}^{n} \frac{\underline{x}_3^m}{\underline{(-x^m)}_2} = \prod_{n=1}^{\infty} \underline{x}_n. \tag{4.2}$$

Another illustration is provided by $u_n(x)$ [4], the recurrence relation for which can be written in the form

$$u_{n+1}(x) = 2xu_n(x) - q^{1-n}\underline{(q^2)}_n u_{n-1}(x),\ n > 0, \tag{4.3}$$

with initial conditions $u_0(x) = 1,\ u_1(x) = 2x, 0 < q < 1,$ and which in the context of special functions generalizes the Hermite polynomials in that

$$\lim_{q \to 1^-} u_n(x) = H_n(x).$$

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