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SOME FERMATIAN INVERSION FORMULAE

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Abstract:
This paper considers some q-extensions of binomial coefficients formed from rising factorial coefficients. Some of the results are applied to a Möbius Inversion Formula and an exponential based on extensions of ideas initially developed by Leonard Carlitz.

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q-series, Fermatian functions, binomial coefficients, Möbius function, rising factorials, Hermite polynomials.

1. Introduction

We shall extend the results in [5,6] to analogous function expressed in terms of Fermatian numbers. We can define [1,2] the n-th reduced Fermatian number in terms of

$$\underline{q}_n = \begin{cases} -q^n \underline{q}_{-n} & (n<0) \\ 1 & (n=0) \\ 1+q+q^2+\ldots+q^{n-1} & (n>0) \end{cases} \tag{1.1}$$

so that

$$\underline{1}_n = n,$$

and

$$\underline{1}_n! = n!,$$

where

$$\underline{q}_n! = \underline{q}_n \underline{q}_{n-1} \ldots \underline{q}_1. \tag{1.2}$$

Accordingly, we define

$$E_z(x) = \sum_{n=0}^{\infty} x^n / \underline{z}_n! \tag{1.3}$$

Note that

$$E_1(x) = e^x.$$

We can also define an inverse [4]

$$1 = E(x).E^{-1}(x).$$

We shall use these to develop a Möbius inversion formula analogous to a result of Carlitz [3]

$$G(t) = e^t F(t) \tag{1.4}$$

in which $F(t)$ and $G(t)$ are power series defined below.

2. Some Fermatian Power Series

We define the (formal) powers series

$$F_t = \sum_{r=0}^{\infty} f_r t^r / \underline{z}_r! \tag{2.1}$$

and

$$G_t = \sum_{r=0}^{\infty} g_r t^r / \underline{z}_r! \tag{2.2}$$

where (formally)

$$g_r \quad = \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix} f_j \qquad (n = 1,2,3,...) \tag{2.3}$$

in which

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}} = \frac{(1-q^n)(1-q^{n-1})...(1-q^{n-k+1})}{(1-q)(1-q^2)...(1-q^k)}. \tag{2.4}$$

We can thus define analogs of other classical polynomials. For example, we can define Fermatian extensions of the Hermite polynomials by $H_{nz}(x)$:

$$E_z(xt)E_z(t) = \sum_{n=0}^{\infty} H_{nz}(x) t^n / \underline{z}_n!. \tag{2.5}$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty} H_{nz}(x) t^n \underline{z}_n! \quad &= \sum_{m=0}^{\infty} \tfrac{x^m t^m}{\underline{z}_m!} \sum_{n=0}^{\infty} \tfrac{t^n}{\underline{z}_n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \tfrac{x^k t^n}{\underline{z}_k! \underline{z}_{n-k}!} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{\underline{z}_n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \end{aligned}$$

and so,

$$H_{nz}(x) \quad = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{2.6}$$

3. Inversion Formulae

We know from the Möbius Inversion Formula that

$$g(n) = \sum_{d|n} f(d) \qquad (n = 1,2,3,...) \qquad (3.1)$$

is equivalent to

$$f(n) = \sum_{cd=n} \mu(c) g(d), \qquad (n = 1,2,3,...) \qquad (3.2)$$

in which $\mu(n)$ is the Möbius function

$$\mu(n) = \begin{cases} (-1)^r & if\ each\ n_i = 1, \\ 1 & if\ each\ n_i = 0, \end{cases}$$

where

$$n = \pm \prod_{i=1}^{r} p_i^{n_i} .$$

It can be verified by a proof similar to the one which appears shortly and in [15] that (3.1) and (3.2) reduce to

$$g_r = \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix} f_j \qquad (r = 0,1,2,3,...) \qquad (3.3)$$

and

$$f_r = \sum_{j=0}^{r} (-1)^{r-j} \begin{bmatrix} r \\ j \end{bmatrix} g_j \qquad (r = 0,1,2,3,...) \qquad (3.4)$$

respectively. We now establish an analog of (1.4):

$$\begin{aligned} G_t &= \sum_{r=0}^{\infty} g_r t^r / \underline{z}_r ! \\ &= \sum_{r=0}^{\infty} \sum_{j=0}^{r} \frac{f_j t^r}{\underline{z}_j ! \underline{z}_{r-j} !} \\ &= \sum_{r=0}^{\infty} f_r \frac{t^r}{\underline{z}_r !} \sum_{k=0}^{\infty} t^k / \underline{z}_k ! \\ &= E_z(t) F_t , \end{aligned}$$

as required.

4. Conclusion

Unless an inverse rising factorial exponential can be established it is unlikely that analogous relationships can be found for the rising factorial except for the trivial case when $f_n = g_n$.

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