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On some Pascal's like triangles. Part 3

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In a series of papers, starting with [1, 2], we discuss new types of Pascal's like triangles. Triangles of the present form, but not with the present sinse, are described in different publications, e.g. [3, 5, 6], but at least the author had not found a research with similar idea.

In the first part of our research we studied properties of some standard sequences and in the second part – of some "special" sequences. Now, we shall construct $(0, 1)$-analogous of the Pascal's like triangles (or "(mod 2)-triangles") from the both previous papers, i.e., we will construct (mod 2)-values of their elements and will discuss the obtained configurations. We will call the new triangles "(0,1)-triangles".

We shall construct the new infinite triangles following their order in [1, 2]. From triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 1 & \mathbf{2} & 1 & & & & & \\
 & & & & 1 & 2 & \mathbf{4} & 2 & 1 & & & & \\
 & & & 1 & 2 & 4 & \mathbf{8} & 4 & 2 & 1 & & & \\
 & & 1 & 2 & 4 & 8 & \mathbf{16} & 8 & 4 & 2 & 1 & & \\
 & 1 & 2 & 4 & 8 & 16 & \mathbf{32} & 16 & 8 & 4 & 2 & 1 & \\
1 & 2 & 4 & 8 & 16 & 32 & \mathbf{64} & 32 & 16 & 8 & 4 & 2 & 1 \\
 & & & & . & & . & & . & & & &
\end{array}
$$

we obtain the corresponding (0,1)-triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 1 & \mathbf{0} & 1 & & & & \\
 & & & 1 & 0 & \mathbf{0} & 0 & 1 & & & \\
 & & 1 & 0 & 0 & \mathbf{0} & 0 & 0 & 1 & & \\
 & 1 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 1 & \\
1 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 1 \\
 & & & . & & . & & . & & &
\end{array}
$$

Obviously, only the elements of the two boundary diagonals have value 1.

Now, for the second triangle:

$$
\begin{array}{c}
\mathbf{1} \\
2 \quad \mathbf{3} \quad 2 \\
4 \quad 6 \quad \mathbf{9} \quad 6 \quad 4 \\
8 \quad 12 \quad 18 \quad \mathbf{27} \quad 18 \quad 12 \quad 8 \\
16 \quad 24 \quad 36 \quad 54 \quad \mathbf{81} \quad 54 \quad 36 \quad 24 \quad 16 \\
32 \quad 48 \quad 72 \quad 108 \quad 162 \quad \mathbf{243} \quad 162 \quad 108 \quad 72 \quad 48 \quad 32 \\
64 \quad 96 \quad 144 \quad 216 \quad 324 \quad 486 \quad \mathbf{729} \quad 486 \quad 324 \quad 216 \quad 144 \quad 96 \quad 64 \\
. \qquad . \qquad .
\end{array}
$$

we have the corresponding (0,1)-triangle

$$
\begin{array}{c}
\mathbf{1} \\
0 \quad \mathbf{1} \quad 0 \\
0 \quad 0 \quad \mathbf{1} \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad \mathbf{1} \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad \mathbf{1} \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \mathbf{1} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
. \qquad . \qquad .
\end{array}
$$

where, only the terms of the middle column are “1”.

For the triangle that elements of the middle column are the consequential natural numbers:

$$
\begin{array}{c}
\mathbf{1} \\
1 \quad \mathbf{2} \quad 1 \\
0 \quad 1 \quad \mathbf{3} \quad 1 \quad 0 \\
0 \quad 0 \quad 1 \quad \mathbf{4} \quad 1 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 1 \quad \mathbf{5} \quad 1 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 1 \quad \mathbf{6} \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \\
. \qquad . \qquad .
\end{array}
$$

we obtain the corresponding (0,1)-triangle

$$
\begin{array}{c}
\mathbf{1} \\
1 \quad \mathbf{0} \quad 1 \\
0 \quad 1 \quad \mathbf{1} \quad 1 \quad 0 \\
0 \quad 0 \quad 1 \quad \mathbf{0} \quad 1 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 1 \quad \mathbf{1} \quad 1 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 1 \quad \mathbf{0} \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \\
. \qquad . \qquad .
\end{array}
$$

Obviously, the same (0,1)-triangle we will obtain for the triangles that elements are the n-th powers of the consequtive natural numbers.

For triangle from [2]

$$
\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 2 & \mathbf{3} & 2 & & & & \\
 & & & 3 & 5 & \mathbf{8} & 5 & 3 & & & \\
 & & 1 & 4 & 9 & \mathbf{17} & 9 & 4 & 1 & & \\
 & 2 & 3 & 7 & 16 & \mathbf{33} & 16 & 7 & 3 & 2 & \\
3 & 5 & 8 & 15 & 31 & \mathbf{64} & 31 & 15 & 8 & 5 & 3 \\
 & & & . & & . & & . & & &
\end{array}
$$

we have the corresponding (0,1)-triangle

$$
\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 0 & \mathbf{1} & 0 & & & & \\
 & & & 1 & 1 & \mathbf{0} & 1 & 1 & & & \\
 & & 1 & 0 & 1 & \mathbf{1} & 1 & 0 & 1 & & \\
 & 0 & 1 & 1 & 0 & \mathbf{1} & 0 & 1 & 1 & 0 & \\
1 & 1 & 0 & 1 & 1 & \mathbf{0} & 1 & 1 & 0 & 1 & 1 \\
 & & & . & & . & & . & & &
\end{array}
$$

while for triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & & & \mathbf{1} & & \\
 & & & & & & & & & 2 & \mathbf{3} & 2 & \\
 & & & & & & & & 3 & 5 & \mathbf{8} & 5 & ... \\
 & & & & & & & 2 & 5 & 10 & \mathbf{18} & 10 & .... \\
 & & & & & & 1 & 3 & 8 & 18 & \mathbf{36} & 18 & ... \\
 & & & & & 2 & 3 & 6 & 14 & 32 & \mathbf{68} & 32 & ... \\
 & & & & 3 & 5 & 8 & 14 & 28 & 60 & \mathbf{128} & 60 & ... \\
 & & & 2 & 5 & 10 & 18 & 32 & 60 & 128 & \mathbf{248} & 128 & ... \\
 & & 1 & 3 & 8 & 18 & 36 & 68 & 128 & 248 & \mathbf{496} & 248 & ... \\
 & 2 & 3 & 6 & 14 & 32 & 68 & 136 & 264 & 512 & \mathbf{1008} & 512 & ... \\
3 & 5 & 8 & 14 & 28 & 60 & 128 & 264 & 528 & 1040 & \mathbf{2048} & 1040 & ... \\
 & & & & & & . & & . & & . & &
\end{array}
$$

the corresponding (0,1)-triangle is

$$
\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 0 & \mathbf{1} & 0 & & & & \\
 & & & 1 & 1 & \mathbf{0} & 1 & 1 & & & \\
 & & 0 & 1 & 0 & \mathbf{0} & 0 & 1 & 0 & & \\
 & 1 & 1 & 0 & 0 & \mathbf{0} & 0 & 0 & 1 & 1 & \\
0 & 1 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 1 & 0 \\
 & . & & . & & . & & & & &
\end{array}
$$

It can be seen easily that we will obtain the same (0,1)-triangle from triangle

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 2 & \mathbf{3} & 2 & & & & \\
 & & & 3 & 5 & \mathbf{8} & 5 & 3 & & & \\
 & & 4 & 7 & 12 & \mathbf{20} & 12 & 7 & 4 & & \\
 & 5 & 9 & 16 & 28 & \mathbf{48} & 28 & 16 & 9 & 5 & \\
6 & 11 & 20 & 36 & 64 & \mathbf{112} & 64 & 36 & 20 & 11 & 6 \\
 & & & . & & . & & . & & &
\end{array}$$

As we have shown in [2], the Fibonacci sequence, generates the following infinite triangle:

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{0} & & & & & \\
 & & & & 1 & \mathbf{1} & 1 & & & & \\
 & & & 1 & 2 & \mathbf{3} & 2 & 1 & & & \\
 & & 2 & 3 & 5 & \mathbf{8} & 5 & 3 & 2 & & \\
 & 3 & 5 & 8 & 13 & \mathbf{21} & 13 & 8 & 5 & 3 & \\
5 & 8 & 13 & 21 & 34 & \mathbf{55} & 34 & 21 & 13 & 8 & 5 \\
 & & & . & & . & & . & & &
\end{array}$$

that has as (0,1)-analogous the triangle

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{0} & & & & & \\
 & & & & 1 & \mathbf{1} & 1 & & & & \\
 & & & 1 & 0 & \mathbf{1} & 0 & 1 & & & \\
 & & 0 & 1 & 1 & \mathbf{0} & 1 & 1 & 0 & & \\
 & 1 & 1 & 0 & 1 & \mathbf{1} & 1 & 0 & 1 & 1 & \\
1 & 0 & 1 & 1 & 0 & \mathbf{1} & 0 & 1 & 1 & 0 & 1 \\
 & & & . & & . & & . & & &
\end{array}$$

As we mentioned in [2], Jacobsthal sequence (see, e.g., [4, 7]) is defined by the recurrence relations:

$$J_0 = 0, J_1 = 1, J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2.$$

Its first numbers are

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, ...$$

When the members of this sequence lie on the boundary diagonals, we obtain the following infinite triangle:

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{0} & & & & & \\
 & & & & 1 & \mathbf{1} & 1 & & & & \\
 & & & 1 & 2 & \mathbf{3} & 2 & 1 & & & \\
 & & 3 & 4 & 6 & \mathbf{9} & 6 & 4 & 3 & & \\
 & 5 & 8 & 12 & 18 & \mathbf{27} & 18 & 12 & 8 & 5 & \\
11 & 16 & 24 & 36 & 54 & \mathbf{81} & 54 & 36 & 24 & 16 & 11 \\
 & & & . & & . & & . & & &
\end{array}$$

The corresponding (0,1)-triangle has the form

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{0} & & & & & \\
 & & & & 1 & \mathbf{1} & 1 & & & & \\
 & & & 1 & 0 & \mathbf{1} & 0 & 1 & & & \\
 & & 1 & 0 & 0 & \mathbf{1} & 0 & 0 & 1 & & \\
 & 1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 1 & \\
1 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 1 \\
 & & & . & & . & & . & & &
\end{array}$$

As we mentioned in [2], Jacobsthal-Lucas sequence (see, e.g., [4, 7]) is defined by the recurrence relations:

$$j_2 = 2, j_1 = 1, j_n = j_{n-1} + 2j_{n-2} \text{ for } n \geq 2.$$

Its first numbers are

$$2, 1, 5, 7, 17, 31, 65, 127, 255, 511, 1025, ...$$

When the members of this sequence lie on the boundary diagonals, we obtain the infinite triangle:

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{2} & & & & & \\
 & & & & 1 & \mathbf{3} & 1 & & & & \\
 & & & 5 & 6 & \mathbf{9} & 6 & 5 & & & \\
 & & 7 & 12 & 18 & \mathbf{27} & 18 & 12 & 7 & & \\
 & 17 & 24 & 36 & 54 & \mathbf{81} & 54 & 36 & 24 & 17 & \\
31 & 48 & 72 & 108 & 162 & \mathbf{243} & 162 & 108 & 72 & 48 & 31 \\
 & & & . & & . & & . & & &
\end{array}$$

Therefore, we obtain a new triangle that is very similar to the triangle from [1]. Its (0,1)-triangle coincides with the previous one.

When we want to obtain in the middle column the triangular numbers, the elements of the left and right diagonals must be 1,2,1,0,0,0,..., i.e.

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 2 & \mathbf{3} & 2 & & & & \\
 & & & 1 & 3 & \mathbf{6} & 3 & 1 & & & \\
 & & 0 & 1 & 4 & \mathbf{10} & 4 & 1 & 0 & & \\
 & 0 & 0 & 1 & 5 & \mathbf{15} & 5 & 1 & 0 & 0 & \\
0 & 0 & 0 & 1 & 6 & \mathbf{21} & 6 & 1 & 0 & 0 & 0 \\
 & & & & . & . & . & & & &
\end{array}$$

The new infinite triangle has the following (0,1)-analogous:

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 0 & \mathbf{1} & 0 & & & & \\
 & & & 1 & 1 & \mathbf{0} & 1 & 1 & & & \\
 & & 0 & 1 & 0 & \mathbf{0} & 0 & 1 & 0 & & \\
 & 0 & 0 & 1 & 1 & \mathbf{1} & 1 & 1 & 0 & 0 & \\
0 & 0 & 0 & 1 & 0 & \mathbf{1} & 0 & 1 & 0 & 0 & 0 \\
 & & & . & & . & & . & & &
\end{array}$$

As we shown in [2], the middle column contains the square numbers, when the elements of the left and right diagonals are 1,3,2,0,0,0,..., while if we want to obtain in the middle column the pentagonal numbers, the elements of the left and right diagonals must be 1,4,3,0,0,0,..., i.e.

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 4 & \mathbf{5} & 4 & & & & \\
 & & & 3 & 7 & \mathbf{12} & 7 & 3 & & & \\
 & & 0 & 3 & 10 & \mathbf{22} & 10 & 3 & 0 & & \\
 & 0 & 0 & 3 & 13 & \mathbf{35} & 13 & 3 & 0 & 0 & \\
0 & 0 & 0 & 3 & 16 & \mathbf{51} & 16 & 3 & 0 & 0 & 0 \\
 & & & . & & . & & . & & &
\end{array}$$

Its (0,1)-analogous is

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 0 & \mathbf{1} & 0 & & & & \\
 & & & 1 & 1 & \mathbf{0} & 1 & 1 & & & \\
 & & 0 & 1 & 0 & \mathbf{0} & 0 & 1 & 0 & & \\
 & 0 & 0 & 1 & 1 & \mathbf{1} & 1 & 1 & 0 & 0 & \\
0 & 0 & 0 & 1 & 0 & \mathbf{1} & 0 & 1 & 0 & 0 & 0 \\
 & & & . & & . & & . & & &
\end{array}$$

Finally, we will mention that the elements of all new constructed (0,1)-triangles are obtained by the formulae from [1] but in the following form.

Let the elements of the above infinite triangles be

$$\begin{array}{ccccccc}
 & & & a_{1,1} & & & \\
 & & a_{2,1} & a_{2,2} & a_{2,3} & & \\
 & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\
 & . & & . & & . &
\end{array}$$

where $a_{i,j}$ are arbitrary real (complex) numbers and for every natural number i and
1. for every natural number j for which $2 \leq j \leq i$ it will be valid:

$$a_{i,j} \equiv a_{i,j-1} + a_{i-1,j-1} (\text{mod} 2);$$

2. for every natural number j for which $i \leq j \leq 2i - 1$ it will be valid:

$$a_{i,j} \equiv a_{i,j+1} + a_{i-1,j-1} (\text{mod} 2).$$

References

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