NNTDM 13 (2007), 2, 15-18

SOME *q*-SERIES INVERSION FORMULAE

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Abstract:

This paper considers some *q*-extensions of binomial coefficients formed from rising factorial coefficients. Some of the results are applied to a Möbius Inversion Formula based on extensions of ideas initially developed by Leonard Carlitz.

Keywords:
q-series, binomial coefficients, Möbius function, rising factorials, exponential functions

1. Introduction

We know from the Möbius Inversion Formula that

$$g(n) = \sum_{d|n} f(d) \qquad (n = 1,2,3,...) \qquad (1.1)$$

is equivalent to

$$f(n) = \sum_{cd=n} \mu(c) g(d), \qquad (n = 1,2,3,...) \qquad (1.2)$$

in which $\mu(n)$ is the Möbius function

$$\mu(n) = \begin{cases} (-1)^r & \textit{if each } n_i = 1, \\ 1 & \textit{if each } n_i = 0, \end{cases}$$

where

$$n = \pm \prod_{i=1}^{r} p_i^{n_i} .$$

If we now take

$$n = \prod_{i=1}^{r} p_i .$$

where the p_j are distinct primes, then it can be verified by a proof similar to the one which appears shortly that (1.1) and (1.2) reduce to

$$g_r = \sum_{j=0}^{r} \binom{r}{j} f_j \qquad (r = 0,1,2,3,...) \qquad (1.3)$$

and

$$f_r = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} g_j \qquad (r = 0,1,2,3,...) \tag{1.4}$$

respectively, where for brevity we put

$$f_r = f(p_1 p_2 ... p_r),$$

and

$$g_r = g(p_1 p_2 ... p_r).$$

Leonard Carlitz generalized many results by considering q-series as analogues of factorials, and from them he constructed a q-series analogue of the ordinary binomial coefficient. In that spirit we shall determine q-series results analogous to (1.3) and (1.4)

2. *q*-series

Carlitz has used q-series in numerous papers; for example [1,2,3,4,5,6,7]. More recently, T. Kim and his colleagues have extended some elegant results in both analytic and elementary number theory with such series in a sequence of papers [10,11,12]. The q-series are defined basically by

$$(q)_n = (1-q)(1-q^2)...(1-q^n), \tag{2.1}$$

with $(q)_0 = 1$. Arising out of these are the so-called q-binomial coefficients which are analogous to ordinary binomial coefficients. Their simplest definition is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}} \tag{2.2}$$

$$= \frac{(1-q^n)(1-q^{n-1})...(1-q^{n-k+1})}{(1-q)(1-q^2)...(1-q^k)}.$$

We now define (formally)

$$g_r = \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix} f_j \qquad (n = 1,2,3,...) \tag{2.3}$$

and we seek to express f_r in terms of g_j.

3. Preliminary Results

Firstly, we use the result

$$\prod_{n=0}^{\infty} \frac{1-ax^n z}{1-x^n z} = \sum_{n=0}^{\infty} \frac{(a)_n}{(x)_n} z^n, \tag{3.1}$$

in which

$$(a)_n \quad (1-a)(1-ax)...(1-ax^{n-1})$$

and $(a)_0 = 1$. This is very similar to an expansion due to Cauchy [9] and noted by Carlitz [9]. If we put a = 0 in (3.1), we get

$$\prod_{n=0}^{\infty}\left(1-x^n z\right)^{-1} = \sum_{n=0}^{\infty} z^n \big/ (x)_n, \qquad (3.2)$$

and if we put $a = x^{-k}$ and replace z by $-x^k z$ we obtain

$$\prod_{n=0}^{k-1}\left(1+x^n z\right) = \sum_{n=0}^{k} \begin{bmatrix} k \\ n \end{bmatrix} x^{\frac{1}{2}n(n-1)} z^n$$

which, for $|x| < 1$, yields

$$\prod_{n=0}^{\infty}\left(1+x^n z\right) = \sum_{n=0}^{\infty} x^{\frac{1}{2}n(n-1)} z^n / (x)_n. \qquad (3.3)$$

4. Relevant Exponential Functions

Carlitz [8] has defined a relevant exponential function, namely,

$$e(z) = \sum_{n=0}^{\infty} z^n (x)_n$$

and so, from (3.2),

$$e(z) = \prod_{n=0}^{\infty}\left(1-x^n z\right)^{-1}$$

and

$$\begin{aligned}(e(z))^{-1} &= \prod_{n=0}^{\infty}\left(1-x^n z\right) \\ &= \sum_{n=0}^{\infty} (-1)^n x^{\frac{1}{2}n(n-1)} z^n / (x)_n .\end{aligned}$$

Then, from the definition of g_r we get

$$\begin{aligned}\sum_{r=0}^{\infty} g_r z^r / (x)_r &= \sum_{r=0}^{\infty}\sum_{j=0}^{r} f_j z^r / (x)_j (x)_{r-j} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{(x)_n} \sum_{=0}^{\infty} f_r z^r / (x)_r \\ &= e(z) \sum_{r=0}^{\infty} f_r z^r / (x)_r .\end{aligned}$$

If we multiply each side by $(e(z))^{-1}$ where $(e(z))^{-1}$ is such that

$$1 = (e(z)) \times (e(z))^{-1},$$

we find that

$$\begin{aligned}
\sum_{r=0}^{\infty} f_r z^r/(x)_r &= (e(z))^{-1} \sum_{r=0}^{\infty} g_r z^r/(x)_r \\
&= \sum_{n=0}^{\infty} (-1)^n x^{\frac{1}{2}n(n-1)} z^n/(x)_n \sum_{r=0}^{\infty} g_r z^r/(x)_r \\
&= \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{r-j} x^{\frac{1}{2}(r-j)(r-j-1)} g_j z^r}{(x)_{r-j}(x)_j} \\
&= \sum_{r=0}^{\infty} \sum_{j=0}^{r} (-1)^{r-j} \begin{bmatrix} r \\ j \end{bmatrix} x^{\frac{1}{2}(r-j)(r-j-1)} g_j z^r/(x)_r.
\end{aligned}$$

Whence,

$$f_r = \sum_{j=0}^{r} (-1)^{r-j} \begin{bmatrix} r \\ j \end{bmatrix} x^{\frac{1}{2}(r-j)(r-j-1)} g_j \qquad (4.1)$$

which is the main result, and which is the required analogue of (1.4).

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AMS Classification Numbers: 11B65, 11B39, 05A30