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On some Pascal's like triangles. Part 2

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In a series of papers, starting with [1], we discuss new types of Pascal's like triangles. Triangles from the present form, but not with the present sense, are described in different publications, e.g. [2, 4, 6], but at least the author had not found a research with similar idea.

In the second part of our research we shall study properties of some "special" sequences.

Here we shall construct new Pascal's like triangles, similarly to these from [1].

If for every natural number $i \geq 1$:

$$a_{i,i} = f_{i-1},$$

where $\{f_k\}_{k\leq 0}$ is the Fibonacci sequence, then we obtain the infinite triangle:

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{0} & & & & & \\
 & & & & 1 & \mathbf{1} & 1 & & & & \\
 & & & 1 & 2 & \mathbf{3} & 2 & 1 & & & \\
 & & 2 & 3 & 5 & \mathbf{8} & 5 & 3 & 2 & & \\
 & 3 & 5 & 8 & 13 & \mathbf{21} & 13 & 8 & 5 & 3 & \\
5 & 8 & 13 & 21 & 34 & \mathbf{55} & 34 & 21 & 13 & 8 & 5 \\
 & & & . & & . & & . & & &
\end{array}$$

Therefore, the terms that stay in the middle column are exactly elements of sequence $\{f_{2n}\}_{n\geq 0}$.

More curious is the triangle, when the terms of the middle column are exactly terms of Fibonacci sequence.

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{0} & & & & & \\
 & & & & 1 & \mathbf{1} & 1 & & & & \\
 & & & -1 & 0 & \mathbf{1} & 0 & -1 & & & \\
 & & 2 & 1 & 1 & \mathbf{2} & 1 & 1 & 2 & & \\
 & -3 & -1 & 0 & 1 & \mathbf{3} & 1 & 0 & -1 & -3 & \\
5 & 2 & 1 & 1 & 2 & \mathbf{5} & 2 & 1 & 1 & 2 & 5 \\
 & & & . & & . & & . & & &
\end{array}$$

We see that the terms of the boundary diagonals the left and the right diagonal of the triangle) are exacly the terms of the Fibonacci sequence that can stay leftly than the first term ($f_0 = 0$) of the standard sequence, i.e., the elements of sequence $\{f_n\}_{n \leq 0}$ (cf., e.g., [5]).

Now, let us construct a new infinite triangle with a middle column coinciding with each of the boundary diagonals of the latter triangle. Then we obtain triangle

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{0} & & & & & \\
 & & & & 1 & \mathbf{1} & 1 & & & & \\
 & & & -3 & -2 & \mathbf{-1} & -2 & -3 & & & \\
 & & 8 & 5 & 3 & \mathbf{2} & 3 & 5 & 8 & & \\
 & -21 & -13 & -8 & -5 & \mathbf{-3} & -5 & -8 & -13 & -21 & \\
55 & 34 & 21 & 13 & 8 & \mathbf{5} & 8 & 13 & 21 & 34 & 55 \\
 & & & . & & . & & . & & &
\end{array}$$

Threfore, the terms that stay in the new boundary diagonals are the elements of sequence $\{f_{2n}\}_{n \leq 0}$ that can stay leftly than the first term of the standard Fibonacci sequence.

Lucas sequence has similar behaviour. When the terms of this sequence stay in the middle column we have

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{2} & & & & & \\
 & & & & -1 & \mathbf{1} & -1 & & & & \\
 & & & 3 & 2 & \mathbf{3} & 3 & 3 & & & \\
 & & -4 & -1 & 1 & \mathbf{4} & 1 & -1 & -4 & & \\
 & -7 & 3 & 2 & 3 & \mathbf{7} & 3 & 2 & 3 & -7 & \\
-11 & -4 & -1 & 1 & 4 & \mathbf{11} & 4 & 1 & -1 & -4 & -11 \\
 & & & . & & . & & . & & &
\end{array}$$

and when they are in the boundary columns we have:

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{2} & & & & & \\
 & & & & 1 & \mathbf{3} & 1 & & & & \\
 & & & 3 & 4 & \mathbf{7} & 4 & 3 & & & \\
 & & 4 & 7 & 11 & \mathbf{18} & 11 & 7 & 4 & & \\
 & 7 & 11 & 18 & 29 & \mathbf{47} & 29 & 18 & 11 & 7 & \\
11 & 18 & 29 & 47 & 76 & \mathbf{123} & 76 & 47 & 29 & 18 & 11 \\
 & & & . & & . & & . & & &
\end{array}$$

Jacobsthal sequence (see, e.g., [3, 7]) is defined by the recurrence relations:

$$J_0 = 0, J_1 = 1, J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2.$$

Its first numbers are

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, ...$$

When the terms of this sequence lie on the boundary diagonals, we obtain the following infinite triangle:

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{0} & & & & & \\
 & & & & 1 & \mathbf{1} & 1 & & & & \\
 & & & 1 & 2 & \mathbf{3} & 2 & 1 & & & \\
 & & 3 & 4 & 6 & \mathbf{9} & 6 & 4 & 3 & & \\
 & 5 & 8 & 12 & 18 & \mathbf{27} & 18 & 12 & 8 & 5 & \\
11 & 16 & 24 & 36 & 54 & \mathbf{81} & 54 & 36 & 24 & 16 & 11 \\
 & & & . & & . & & . & & &
\end{array}$$

It is interesting to compare the present triangle with the second triangle from [1]:

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 2 & \mathbf{3} & 2 & & & & & \\
 & & & & 4 & 6 & \mathbf{9} & 6 & 4 & & & & \\
 & & & 8 & 12 & 18 & \mathbf{27} & 18 & 12 & 8 & & & \\
 & & 16 & 24 & 36 & 54 & \mathbf{81} & 54 & 36 & 24 & 16 & & \\
 & 32 & 48 & 72 & 108 & 162 & \mathbf{243} & 162 & 108 & 72 & 48 & 32 & \\
64 & 96 & 144 & 216 & 324 & 486 & \mathbf{729} & 486 & 324 & 216 & 144 & 96 & 64 \\
 & & & & . & & . & & . & & & &
\end{array}$$

Jacobsthal-Lucas sequence (see, e.g., [3, 7]) is defined by the recurrence relations:

$$j_2 = 2, j_1 = 1, j_n = j_{n-1} + 2j_{n-2} \text{ for } n \geq 2.$$

Its first numbers are

$$2, 1, 5, 7, 17, 31, 65, 127, 255, 511, 1025, ...$$

When the terms of this sequence lie on the boundary diagonals, we obtain the triangle:

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{2} & & & & & \\
 & & & & 1 & \mathbf{3} & 1 & & & & \\
 & & & 5 & 6 & \mathbf{9} & 6 & 5 & & & \\
 & & 7 & 12 & 18 & \mathbf{27} & 18 & 12 & 7 & & \\
 & 17 & 24 & 36 & 54 & \mathbf{81} & 54 & 36 & 24 & 17 & \\
31 & 48 & 72 & 108 & 162 & \mathbf{243} & 162 & 108 & 72 & 48 & 31 \\
 & & & . & & . & & . & & &
\end{array}$$

Therefore, we obtain a new infinite triangle that is very similar to the triangle from [1].

As it is well-knowm, the Fibonacci line

$$\{f_n\}_{\infty \leq n + \infty 0} : ..., -3, 2, -1, 1, 0, 1, 1, 2, 3, ...,$$

has the property:

$$|f_{-n}| = f_n$$

for every natural number $n \geq 0$. This property is not valid for Jacobsthal and Jacobsthal-Lucas sequences, but we can construct two new sequences that can call respective pseudo-Jacobsthal line and pseudo-Jacobsthal-Lucas line with the forms:

$$\{J_{2n}\}_{\infty \leq n + \infty 0} : ..., -5, 3, -1, 1, 0, 1, 1, 3, 5, ...$$

and

$$\{j_{2n}\}_{\infty \leq n + \infty 0} : ..., -7, 5, -1, 2, 1, 5, 7, 17, ...$$

respectively, i.e.,

$$J_{-n} = (-1)^{n+1} J_n$$

and

$$j_{-n} = (-1)^n j_n.$$

Now, we see that if we construct a triangle with a middle column coinciding with the Jacobsthal sequence, then its boundary columns will be the elements that stay on the left side of $J_0 (= 0)$ in the pseudo-Jacobsthal line:

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{0} & & & & & \\
 & & & & 1 & \mathbf{1} & 1 & & & & \\
 & & & -1 & 0 & \mathbf{1} & 0 & -1 & & & \\
 & & 3 & 2 & 2 & \mathbf{3} & 2 & 2 & 3 & & \\
 & -5 & -2 & 0 & 2 & \mathbf{5} & 2 & 0 & -2 & -5 & \\
11 & 6 & 4 & 4 & 6 & \mathbf{11} & 6 & 4 & 4 & 6 & 11 \\
 & & & . & & . & & . & & &
\end{array}$$

The same property will be valid for the Jacobsthal-Lucas sequence and pseudo-Jacobsthal-Lucas line:

$$\begin{array}{ccccccccc}
 & & & & \mathbf{2} & & & & \\
 & & & -1 & \mathbf{1} & 1 & & & \\
 & & 5 & 4 & \mathbf{5} & 4 & 5 & & \\
 & -7 & -1 & 2 & \mathbf{7} & 2 & -2 & -7 & \\
17 & 10 & 8 & 10 & \mathbf{17} & 10 & 8 & 10 & 17 \\
 & & . & & . & & . & &
\end{array}$$

When we want to obtain in the middle column the triangular numbers, the elements of the left and right diagonals must be 1,2,1,0,0,0,..., i.e.

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & 2 & \mathbf{3} & 2 & & & & \\
 & & & 1 & 3 & \mathbf{6} & 3 & 1 & & & \\
 & & 0 & 1 & 4 & \mathbf{10} & 4 & 1 & 0 & & \\
 & 0 & 0 & 1 & 5 & \mathbf{15} & 5 & 1 & 0 & 0 & \\
0 & 0 & 0 & 1 & 6 & \mathbf{21} & 6 & 1 & 0 & 0 & 0 \\
 & & & & . & . & . & & & &
\end{array}$$

As we saw above, the middle column contains the square numbers, when the elements of the left and right diagonals are 1,3,2,0,0,0,..., while if we like to obtain in the middle column the pentagonal numbers, the elements of the left and right diagonals must be 1,4,3,0,0,0,..., i.e. we have the infinite triangle

					1					
				4	**5**	4				
			3	7	**12**	7	3			
		0	3	10	**22**	10	3	0		
	0	0	3	13	**35**	13	3	0	0	
0	0	0	3	16	**51**	16	3	0	0	0
			.		.		.			

Finally, if we like to obtain in the middle column the n-gonal numbers, the elements of the left and right diagonals must be $1, n-1, n-2, 0, 0, 0, ...$

References

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