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INTEGER STRUCTURE ANALYSIS OF PRIMES AND COMPOSITES FROM SUMS OF TWO FOURTH POWERS

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Abstract

An integer structure (IS) analysis of the sum $(x^4 + y^4)$ is done using the modular ring Z_6. This sum generates many primes and the row structure of such primes is explored. The class functions of the composite factors of this sum are also given, and these, together with the associated row functions, illustrate why it is impossible to produce an integer to the fourth power from such sums. The overall results are consistent with those previously found with IS analysis.

1. Introduction

Sums of even powers, *m*, can produce primes, as was shown by Fermat with $2^{2^n} + 1^{2^n}$ ($n \le 4$), whereas sums of odd powers may be factored. For example,

$$x^3 + y^3 = (x+y)(x^2 + y^2 - xy) \quad (1.1)$$
$$x^5 + y^5 = (x+y)(x^4 + y^4 - xy(x^2 - xy + y^2)) \quad (1.2)$$
$$x^7 + y^7 = (x+y)(x^6 + y^6 - xy(x^4 - xy(x^2 - xy + y^2) + y^4)). \quad (1.3)$$

However, if $\frac{1}{2}m$ is odd, a factored form can arise; for instance,

$$x^6 + y^6 = (x^2 + y^2)((x^2 + 3xy + y^2)^2 - 6xy(x+y)^2). \quad (1.4)$$

This does not happen for the sum of two squares, those, a matter of related interest, Adler [1] has proved that

$$x^2 + y^2 = 3xy \mp 1$$

generates the Fibonacci numbers with odd and even indices, respectively, by equating *x* to any Fibonacci number with odd (even) index and solving for *y*; the larger root will be the Fibonacci number with the next larger odd (even) index.

More particularly, $(x^4 + y^4)$ produces relatively large numbers of primes and in this paper we shall examine the underlying structure of this sum by using the modular ring Z_6 [2].

Integers in this ring have the form $(6r_i + (i-3))$ in which the row $r_i \in \bar{i}_6$, the class. Even integers $\in \{\bar{1}_6, \bar{3}_6, \bar{5}_6\}$; $\bar{5}_6$ has no even powers, and $3|N \forall N \in \bar{3}_6$. Odd integers $\in \{\bar{2}_6, \bar{4}_6, \bar{6}_6\}$; $\bar{2}_6$ has no even powers, and $3|\, N \,\forall N \in \bar{6}_6$.

2. Primes from $(x^4 + y^4)$

Fermat showed that primes generated by $(4r_1 + 1) \in \bar{1}_4 \subset Z_4$ are equal to a unique sum of squares [3]. In Z_6, these primes fall in $\bar{2}_6$ in odd rows and in $\bar{4}_6$ in even rows [7]. Since

$$x^4 + y^4 = (x^2)^2 + (y^2)^2, \qquad (2.1)$$

the primes formed from Equation (2.1) will be of this type. Table 1 shows that the (x^4, y^4) couple $(\bar{1}_6 \bar{4}_6)$ yields the most primes. This is understandable since $x \in \{\bar{1}_6, \bar{5}_6\}$ and $y \in \{\bar{4}_6, \bar{2}_6\}$ contribute to $(x^4 + y^4)$ which must belong to $\bar{2}_6$ (Table 2). Furthermore, since $\bar{2}_6$ has no even powers there is room for more primes, whereas $3 \mid N \in \bar{6}_6$ so there are no primes in this class.

x↓y→	1	3	5	7	9	11	13	15	17	19	21	23	25	27
2	⊗	⊕	⊗	⊗	⊕	⊗			⊗	⊗		⊗		⊕
4	⊗	⊕	⊗	⊗		⊗	⊗		⊗			⊗		
6	*			*		*				*			*	
8		⊕	⊗		⊕			⊕		⊗		⊗	⊗	
10				⊗	⊕		⊗			⊗	⊕	⊗		
12							*			*			*	
14			⊗				⊗	⊕	⊗	⊗				
16	⊗	⊕	⊗			⊗			⊗		⊕			⊕
18			*	*		*						*		
20	⊗	⊕			⊗				⊗					

Table 1: (*x*,*y*) couples which produce primes (*NB*: $(\bar{5}_6)^4 \in \bar{1}_6; (\bar{2}_6)^4 \in \bar{4}_6$)
Legend: $(x^4 y^4): \bar{1}_6 \bar{4}_6$ ⊗; $\bar{1}_6 \bar{6}_6$ ⊕; $\bar{3}_6 \bar{4}_6$ *

x^4	y^4	$x^4 + y^4$
$\bar{1}_6$	$\bar{4}_6$ $\bar{6}_6$	$\bar{2}_6$ $\bar{4}_6$
$\bar{3}_6$	$\bar{4}_6$ $\bar{6}_6$	$\bar{4}_6$ $\bar{6}_6$

Table 2: Classes for $(x^4 + y^4)$

(Functions for the rows of squares have been given previously [7].)

We may use these functions here to show how the row structure of the primes arises. As noted above, primes in Z_6 equal a sum of squares when they have odd rows in $\bar{2}_6$ and even rows in $\bar{4}_6$ [7]. The (*a*,*b*) couples $(\bar{1}_6, \bar{4}_6)$, $a = x^2, b = y^2$, need only be considered when $3 \nmid N$ (⊗in Table 1). These primes fall in odd rows in $\bar{2}_6$. The component $a = x^2 = 2^m Q$ (Table 3).

Table 3 shows that *m* is even and $Q \in \bar{4}_6$ so that

$$x^4 = a^2 = 2^{2m}(6r_4 + 1)^2 \tag{2.2}$$

and R_1, the row for x^4, is given by

$$R_1 = 2^{2m}\left(6n_2^2 + 2n_2 + \tfrac{1}{6}\right) + \tfrac{1}{3} \tag{2.3}$$

with n_2 equal to the row, r_4, of $Q \in \bar{4}_6$ (Table 3). For example, with *a*=64$= 2^6 \times 1$, *m*=6, n_2=0, so $R_1 = 683$.

x	$a = x^2$	m	Q	Class of Q	r_4, row of Q	R_1, row of a^2
2	4	2	1	$\bar{4}_6$	0	3
4	16	4	1	$\bar{4}_6$	0	43
8	64	6	1	$\bar{4}_6$	0	683
10	100	2	25	$\bar{4}_6$	4	1667
14	196	2	49	$\bar{4}_6$	8	6403
16	256	8	1	$\bar{4}_6$	1	10923
20	400	4	25	$\bar{4}_6$	4	26667

Table 3: $a = x^2 = 2^m Q$

$y^2 = b \in \bar{4}_6 \Rightarrow b^2 \in \bar{4}_6$, and so

$$b^2 = 6R_4 + 1$$

with R_4 given by [7]:

$$R_4 = 2n_2'(3n_2' + 1) \tag{2.4}$$

in which n_2' is the row of *b*. For example, with *y*=19, $n_2' = 60$.

The prime from $a^2 + b^2$ falls in an odd row in $\bar{2}_6$ and

$$\begin{aligned} \bar{1}_6 + \bar{4}_6 &= 6R_1 - 2 + 6R_4 + 1 \\ &= 6R_2 - 1 \end{aligned} \tag{2.5}$$

so that

$$R_1 + R_4 = R_2. \tag{2.6}$$

Thus, the row of the prime is given by

$$r_p = 2^{2m}\left(6n_2^2 + 2n_2 + \tfrac{1}{6}\right) + \tfrac{1}{3} + 2n_2'(3n_2' + 1). \tag{2.7}$$

Some examples for the $(\bar{1}_6\bar{4}_6),(a^2, b^2)$couples (⊗in Table 1) are given in Table 4.

x	a	m	Q	r_4	R_1 (Equation (2.3))
14	196	2	49	8	$2^4\left(6 \times 64 + 2 \times 8 + \tfrac{1}{6}\right) + \tfrac{1}{3} = 6403$

y	b	$n_2^{/}$	R_4 (Eq (2.4)	$R_2 = R_1 + R_4$	Prime $(6R_2 - 1)$
5	25	4	$2\times 4(3\times 4+1)=104$	6507	39041
13	169	28	4760	11163	666977
17	289	48	13920	20323	121937
19	361	60	21720	28123	1688737

Table 4: Some examples for the $(\bar{1}_6 \bar{4}_6),(a^2, b^2)$couples

A similar analysis may be made for the $(\bar{1}_6, \bar{6}_6)$couple. In this case, R_1 is given by Equation (2.3) and R_6 is given by [7].

$$R_6 = 6(q^2 + q) + 1 \tag{2.8}$$

with

$$q = 1 + 12(\tfrac{1}{2} j(j+1)) = 1 + 6r_6(r_6 + 1) \tag{2.9}$$

with r_6 the row of y, and, since

$$\begin{aligned} \bar{1}_6 + \bar{6}_6 &= 6R_1 - 2 + 6R_6 + 3 \\ &= 6(R_1 + R_6) + 1 \\ &= 6R_4 + 1 \end{aligned} \tag{2.10}$$

so that

$$R_1 + R_6 = R_4 \tag{2.11}$$

in which R_4 is the row of the prime in $\bar{4}_6$. For example, with x=16, y=21, $R_1 = 10923$ (Table 3), and with $r_6 = 3, q = 73$ so that $R_6 = 32413$, hence

$$R_4 = 10923 + 32413 = 43336,$$

so that the prime

$$(6R_4 + 1) = 260017.$$

Note that R_4 is even, as expected. Of course, when calculated directly

$$x^4 + y^4 = 260017,$$

but this result gives no information about the underlying row structure.

3. Composites from $x^4 + y^4$

With r_i, r_j the rows of x, y, respectively

$$x^4 + y^4 = n_4(f(r_i^4, r_j^4)) + n_3(f(r_i^3, r_j^3)) + n_2(f(r_i^2, r_j^2)) + n_1(f(r_i, r_j)) + n_0. \tag{3.1}$$

The functions for the various (x, y) couples are given in Table 5. This table is useful for ascertaining factors of $N = x^4 + y^4$. The five terms permit N to be broken up into more easily factored components.

The prime 17 is a common factor and Table 6 shows the factor structure of the various class couples that yield 17 as a factor.

Although $n_0 = 17$ for $(\bar{1}_6, \bar{2}_6),(\bar{1}_6, \bar{4}_6),(\bar{5}_6, \bar{4}_6),(\bar{5}_6, \bar{2}_6)$, this does not give them the highest percentage of 17|N (Table 7). Note that when $n_0 = 17, (x^4 + y^4) \in \bar{2}_6$, so that this sum can never equal a fourth power because this class contains no powers. The remaining class couples $\left(\bar{1}_6,\bar{6}_6\right),\left(\bar{5}_6,\bar{6}_6\right),\left(\bar{3}_6,\bar{2}_6\right),\left(\bar{3}_6,\bar{4}_6\right)\in\bar{4}_6$ and $\left(\bar{3}_6,\bar{6}_6\right)\in\bar{6}_6$ each have $3|x$ and/or $3|y$. Equation (3.1) should also be useful in the analysis of the additive components of primes formed from $\left(x^4+y^4\right)$.

x	**y**	x^4+y^4	$n_4\left(f\left(r_i^4,r_j^4\right)\right)$	$n_3\left(f\left(r_i^3,r_j^3\right)\right)$	$n_2\left(f\left(r_i^2,r_j^2\right)\right)$	$n_1\left(f\left(r_i,r_j\right)\right)$	n_0
$\bar{1}_6$	$\bar{2}_6$	$\bar{2}_6$	$6^4\left(r_1^4+r_2^4\right)$	$-4\times6^3\left(2r_1^3+r_2^3\right)$	$6^3\left(4r_1^2+r_2^2\right)$	$-4\times6\left(8r_1+r_2\right)$	17
$\bar{1}_6$	$\bar{4}_6$	$\bar{2}_6$	$6^4\left(r_1^4+r_4^4\right)$	$-4\times6^3\left(2r_1^3-r_4^3\right)$	$6^3\left(4r_1^2+r_4^2\right)$	$-4\times6\left(8r_1-r_4\right)$	17
$\bar{1}_6$	$\bar{6}_6$	$\bar{4}_6$	$6^4\left(r_1^4+r_6^4\right)$	$-4\times6^3\left(2r_1^3-3r_6^3\right)$	$6^3\left(4r_1^2+9r_6^2\right)$	$-4\times6\left(8r_1-27r_6\right)$	97
$\bar{5}_6$	$\bar{4}_6$	$\bar{2}_6$	$6^4\left(r_5^4+r_4^4\right)$	$4\times6^3\left(2r_5^3+r_4^3\right)$	$6^3\left(4r_5^2+r_4^2\right)$	$4\times6\left(8r_5+r_4\right)$	17
$\bar{5}_6$	$\bar{2}_6$	$\bar{2}_6$	$6^4\left(r_5^4+r_2^4\right)$	$4\times6^3\left(2r_5^3-r_2^3\right)$	$6^3\left(4r_5^2+r_2^2\right)$	$4\times6\left(8r_5-r_2\right)$	17
$\bar{5}_6$	$\bar{6}_6$	$\bar{4}_6$	$6^4\left(r_5^4+r_6^4\right)$	$4\times6^3\left(2r_5^3+3r_6^3\right)$	$6^3\left(4r_5^2+9r_6^2\right)$	$4\times6\left(8r_5+27r_6\right)$	97
$\bar{3}_6$	$\bar{2}_6$	$\bar{4}_6$	$6^4\left(r_3^4+r_2^4\right)$	$-4\times6^3\left(r_2^3\right)$	$6^3\left(r_2^2\right)$	$-4\times6\left(r_2\right)$	1
$\bar{3}_6$	$\bar{4}_6$	$\bar{4}_6$	$6^4\left(r_3^4+r_4^4\right)$	$4\times6^3\left(r_4^3\right)$	$6^3\left(r_4^2\right)$	$4\times6\left(r_4\right)$	1
$\bar{3}_6$	$\bar{6}_6$	$\bar{6}_6$	$6^4\left(r_3^4+r_6^4\right)$	$2\times6^4\left(r_6^3\right)$	$6^3\left(9r_6^2\right)$	$4\times6\left(27r_6\right)$	81

Table 5: Components of Equation (3.1)

x	y	r_i	r_j	x,y classes	Component sums Equation (3.1), Table 5	x^4+y^4	Factors $17\times$
2	13	0	2	$\bar{5}_6\ \bar{4}_6$	$(1+2+3+4)=17\times1680$ $(5)=17$	2857 ($\bar{2}_6$)	41^2
2	21	0	3	$\bar{5}_6\ \bar{6}_6$	$(1+2+3+4+5)=17^2\times673$	194497 ($\bar{4}_6$)	17×673
4	9	1	1	$\bar{1}_6\ \bar{6}_6$	$(2+3)=17\times216$ $(1+4+5)=17\times185$	6817 ($\bar{4}_6$)	401
4	15	1	2	$\bar{1}_6\ \bar{6}_6$	$(1)=17\times6^4$ $(2+3+4+5)=17\times1697$	50881 ($\bar{4}_6$)	41×73
4	19	1	3	$\bar{1}_6\ \bar{4}_6$	$(1+2+3+4)=17\times7680$ $(5)=17\times1$	130577 ($\bar{2}_6$)	7681

4	25	1	4	$\bar{1}_6\ \bar{4}_6$	$(1+2+3+4)=17\times 2^4\times 1437$ $(5)=17\times 1$	390881 ($\bar{2}_6$)	22993
6	3	1	0	$\bar{3}_6\ \bar{6}_6$	$(1+5)=1377=17\times 3^4$ $(2+3+4)=0$	1377 ($\bar{6}_6$)	3^4
6	5	1	1	$\bar{3}_6\ \bar{2}_6$	$(1+2+3=4+5)=17\times 113$	1921 ($\bar{4}_6$)	113
8	1	1	0	$\bar{5}_6\ \bar{4}_6$	$(1+2+3+4)=17\times 240$ $(5)=17\times 1$	4097 ($\bar{2}_6$)	241
8	13	1	2	$\bar{5}_6\ \bar{4}_6$	$(1)=17\times 6^4$ $(2+3+4+5)=17\times 5^4$	32657 ($\bar{2}_6$)	17×113
8	21	1	3	$\bar{5}_6\ \bar{6}_6$	$(3)=17\times 6^3\times 5$ $(1+2+4+5)=17\times 10601$	198577 ($\bar{4}_6$)	11681
10	3	2	0	$\bar{1}_6\ \bar{6}_6$	$(3+5)=17\times 11\times 19$ $(1+2+4)=17\times 3\times 2^7$	10081 ($\bar{4}_6$)	593
12	7	2	1	$\bar{3}_6\ \bar{4}_6$	$(1)=17\times 6^4$ $(2+3+4+5)=17\times 5\times 13$	23137 ($\bar{4}_6$)	1361
12	23	2	4	$\bar{3}_6\ \bar{2}_6$	$(1)=17\times 16\times 6^4$ $(2+3+4+5)=-17\times 5\times 13\times 47$	300577 ($\bar{4}_6$)	17681
12	27	2	4	$\bar{3}_6\ \bar{6}_6$	$(1)=17\times 2^4\times 6^4$ $(2+3+4+5)=17\times 11745$	552177 ($\bar{6}_6$)	$3^4\times 401$
14	23	2	4	$\bar{5}_6\ \bar{2}_6$	$(1+5)=17\times 20737$ $(2+3+4)=-17\times 2016$	318257 ($\bar{2}_6$)	97×193
14	27	2	4	$\bar{5}_6\ \bar{6}_6$	$(1)=17\times 20736$ $(2+3+4+5)=17\times 12785$	569857 ($\bar{4}_6$)	33521
16	9	2	1	$\bar{1}_6\ \bar{6}_6$	$(2)=-17\times 2^5\times 81$ $(1+3+4+5)=17\times 6833$	72097 ($\bar{4}_6$)	4241
16	15	3	2	$\bar{1}_6\ \bar{6}_6$	$(2+5)=-17\times 7^2\times 31$ $(1+3+4)=17\times 2^5\times 3\times 87$	116161 ($\bar{4}_6$)	6833
16	19	3	3	$\bar{1}_6\ \bar{4}_6$	$(1+2+3+4)=17\times 2^8\times 3^2\times 5$ $(5)=17\times 1$	195857 ($\bar{2}_6$)	41×281
16	25	3	4	$\bar{1}_6\ \bar{4}_6$	$(1+2+3+4)=17\times 26832$ $(5)=17\times 1$	456161 ($\bar{2}_6$)	26833

Table 6: Factor structure of the various class couples that yield 17 a s a factor

Class couples	%with factor 17	n_0
$\bar{1}_6\ \bar{2}_6$	0	17
$\bar{1}_6\ \bar{4}_6$	19	17
$\bar{1}_6\ \bar{6}_6$	24	97
$\bar{3}_6\ \bar{2}_6$	9.5	1
$\bar{3}_6\ \bar{4}_6$	5	1
$\bar{3}_6\ \bar{6}_6$	9.5	81
$\bar{5}_6\ \bar{2}_6$	5	17
$\bar{5}_6\ \bar{4}_6$	14	17
$\bar{5}_6\ \bar{6}_6$	14	97

Table 7: Percent with factor 17 from Table 6

The right end digit (RED) of $\left(x^4+y^4\right)^*=1,7;$ the asterisk indicates a RED. Since all odd $\left(N^4\right)^*=1,5$, then $\left(x^4+y^4\right)^*=7$ can never equal z^4. Furthermore, even when $\left(x^4+y^4\right)^*=1,$ if the sum falls in $\bar{2}_6$, then it can never equal z^4, because $\bar{2}_6$ contains no powers. All odd $\left(N^4\right)$ have rows $R_1^*=0,4,N^4\in\bar{4}_6$ and $R_6^*=3,7,N^4\in\bar{6}_6$.

The class couples $\left(\bar{1}_6,\bar{6}_6\right)$ and $\left(\bar{3}_6,\bar{2}_6\right)$, Table 6, that have $\left(x^4+y^4\right)^*=1,$ fall in $\bar{4}_6$ which contains even powers. All the (*x*,*y*) couples in this case have 5|*x* or 5|*y*, as well as 3|*x* or 3|*y*.

The row, R_4, of the $\left(x_4+y^4\right)$ sum, if the sum gives an integer to the fourth power, is [7}:

$$R_4=8K(12K+1) \tag{3.2}$$

with

$$K=\tfrac{1}{2}n(3n+1),$$

n is the row of $\left(x^4+y^4\right)^{\frac{1}{4}}$, or, using the row $n_2^/$ of $\left(x^4+y^4\right)^{\frac{1}{2}}$,

$$R_4=2n_2^/\left(3n_2^/+1\right).$$

(Section 2). Of course, from Fermat's Last Theorem $\left(x^4+y^4\right)\neq z^4$, so *K* can never be an integer. However, one can see that by using integer structure, the analysis can be reduced to a few specific cases where z^4 is possible. Let us consider such cases from Table 6. A composite integer $N\ \in\bar{4}_6$ is given by [5]:

$$N=p^2+6pt, \tag{3.3}$$

where *p* is the lowest prime factor, so that

$$t=\tfrac{1}{6p}\left(N-p^2\right). \tag{3.4}$$

As can be seen from Table 8, $t^* = 0,4,6$. When $N = (p_1 p_2 p_3 \ldots)^4$ or p_1^4, t always has a factor p_1.

N^*	1	3	5	7	9
$(N^4)^*$	1	1	5	1	1
t^* for N^4	0	4	0	6	0
factor p^*	1	3	5	7	9

Table 8: Some factors

The four sums in Table 6 that have a RED=1 and fall in $\bar{4}_6$, and are hence potentially equal to an integer power, have values of *t* incompatible with N^4, that is, *t* has factors $p^/$ with $(p^/)^* \neq 7$ (Table 9); *p**=7 is required when *p*=17 for N^4.

The same type of analysis may be made for $(x^4 + y^4)$combinations that give factors with REDs 1,3,5,9.

$x^4 + y^4$	t	Factors of t
50881	496	$2^4 \times 31$
1921	16	$2^4 \times 1$
10081	96	$2^5 \times 3$
116161	1136	$2^4 \times 71$

Table 9: Factors of *t*

4. Final Comments

Lists of prime factors of $\left(a^{2^m} + b^{2^m}\right)$ with $(a,b) \in \{(2,3),(2,5),(4,5),(6,5)\}$ in the form $(k2^n + 1)$ are available [10]. In the modular ring Z_4 [3], the primes are $(4r_1 + 1) \in \bar{1}_4$ or $(4r_3 + 3) \in \bar{3}_4$ so that $r_1 = k2^{n-2}$ and $r_3 = \frac{1}{2}(k2^{n-1} - 1)$, while for Z_6, when $p \in \bar{2}_6, r_2 = \frac{1}{6}(k2^n + 2)$and for $p \in \bar{4}_6, r_4 = \frac{1}{6}(k2^n)$. An integer solution for the row defines the class. For example, with (*a*,*b*)=(2,3), *m*=3 gives

$$\begin{aligned} 2^8 + 3^8 &= 6817 \\ &= 6 \times 1136 + 1 \\ &= 4 \times 1704 + 1, \textit{ for } Z_4, \end{aligned}$$

with *p*=17, *k*=1, *n*=4 so for $Z_4, r_1 = 4, r_3 = \frac{7}{2}$, so $p \in \bar{1}_4$. For Z_6 $r_2 = 3, r_4 = \frac{8}{3}$ so $p \in \bar{2}_6$ with an odd row.

As has been shown previously [4,6,8,9], integer structure (IS) analysis allows a much wider and in-depth treatment of some numerical systems. In the present case, IS analysis has provided a variety of restraints and functions that illustrate the extreme difficulty of ever finding an integer z such that $z^4 = x^4 + y^4$. Furthermore, the ease of producing a prime from $\left(x^4 + y^4\right)$ can often be more readily assessed with class analysis.

The analyses here are applicable to all even power sums when $\frac{1}{2}n$ is even. However, when $\frac{1}{2}n$ is odd, the sum effectively becomes an odd-powered sum and can be factored so that there will be no primes. For instance,

$$x^6 + y^6 = \left(x^2\right)^3 + \left(y^2\right)^3, \tag{4.1}$$

and, if x^2 and y^2 are substituted for x and y in Equation (1.1) this yields Equation (1.4). Similarly,

$$x^{10} + y^{10} = \left(x^2\right)^5 + \left(y^2\right)^5, \tag{4.2}$$

which can be factored according to Equation (1.2), and so on.

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