NNTDM 13 (2007), 1, 25-30

SOME GENERALIZED RISING BINOMIAL COEFFICIENTS

A. G. Shannon
Warrane College, The University of New South Wales, Kensington 1465, &
RafflesKvB, 99 Mount Street, North Sydney, NSW 2065, Australia
Email: tony@warrane.unsw.edu.au; tonySHANNON@kvb.edu.au

Abstract:
This paper considers some properties of generalized binomial coefficients which are defined in terms of rising factorial coefficients. Analogies with classical results in number theory and some generalized special functions are highlighted.

Keywords:
q-series, binomial coefficients, rising factorials, generalized Fibonacci numbers, Appell set, Gauss-Cayley generalizations.

AMS Classification Numbers: 11B65, 11B39, 05A30

1. Introduction

In a previous paper [6], falling and rising factorials were utilised in the following forms. The falling factorial, an *r*-permutation of *n* distinct objects, is given by

$$\begin{aligned} n^{\underline{r}} &= P(n, r) \\ &= n(n-1)...(n-r+1) \end{aligned} \tag{1.1}$$

and is such that

$$\begin{aligned} \nabla P(n,r) &= P(n,r) - P(n-1,r) \\ &= rP(n-1,r-1), \end{aligned} \tag{1.2}$$

(see, for example Riordan [5]). Similarly, we showed for the rising factorial of *n*

$$n^{\overline{r}} = n(n+1)...(n+r-1) \tag{1.3}$$

that

$$\begin{aligned} \nabla n^{\overline{r}} &= n^{\overline{r}} - (n-1)^{\overline{r}} \\ &= rn^{\overline{r-1}}. \end{aligned} \tag{1.4}$$

This is a recurrence relation for $n^{\overline{r}}$, which is an *r* permutation of *n* + *r* - 1 objects, and which is related to the Stirling numbers. Corresponding binomial coefficients were also considered, namely,

$$\begin{aligned}\binom{n}{r} &= \frac{n(n-1)...(n-r+1)}{r(r-1)...(r-r+1)} \\ &= \frac{n^{\underline{r}}}{r^{\underline{r}}}\end{aligned} \tag{1.5}$$

in which $n^{\underline{r}}$ is the falling r-factorial of n and

$$C(n,r) = \frac{n^{\overline{r}}}{r^{\overline{r}}} \tag{1.6}$$

in which $n^{\overline{r}}$ is the rising r-factorial of n. Thus

$$C(n,r) = \frac{n(n+1)...(n+r-1)}{r(r+1)...(r+r-1)}, \tag{1.7}$$

which is also suggested by the Gauss-Cayley form of the generalized binomial coefficient. Here it is proposed to consider properties of another rising binomial coefficient defined by

$$\left\langle {n \atop k} \right\rangle_a = \frac{n^{\overline{a}}}{k^{\overline{a}}(n-k)^{\overline{a}}}, \tag{1.8}$$

The properties to be developed are analogues of some classical number theory results.

2. Saalschutz' Theorem

Carlitz [2] used the formula of Saalschutz

$$\sum_{k=0}^{n} \frac{(-n)^{\overline{k}} a^{\overline{k}} b^{\overline{k}}}{k! c^{\overline{k}} d^{\overline{k}}} = \frac{(c-a)^{\overline{n}}(c-b)^{\overline{n}}}{c^{\overline{n}}(c-a-b)^{\overline{n}}} \tag{2.1}$$

where

$$c+d = -n+a+b+1,$$

in order to prove some of Macmahon's results [4] and Dixon's theorem

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{k}^3 = \frac{(-1)^n (3n)!}{(n!)^3}.$$

Carlitz also used the q-analogue of Saalschutz' theorem to give an elegant proof of an identity due to Wright [7].

If we let $b = n-1$, then the rising binomial analogue of Saalschutz formula becomes

$$\sum_{k=0}^{n} (-1)^k (n-1)^{\overline{k}} \binom{n}{k} \left\langle {a \atop c} \right\rangle_k = \left\langle {2c-a-n+1 \atop c} \right\rangle_n \Big/ \left\langle {2c-a-n+1 \atop c-a} \right\rangle_n \tag{2.2}$$

Proof: If $b = n - 1$, then $c + d = -n + a + b + 1$ implies that $a = c + d$, and

$$
\begin{aligned}
\sum_{k=0}^{n} \frac{(-n)^{\overline{k}}\, a^{\overline{k}} b^{\overline{k}}}{k!\, c^{\overline{k}} d^{\overline{k}}} &= \sum_{k=0}^{n} \frac{(-n)^{\overline{k}}\, a^{\overline{k}} b^{\overline{k}}}{k!\, c^{\overline{k}} d^{\overline{k}}} \\
&= \sum_{k=0}^{n} \frac{(-n)^{\overline{k}}}{k!} (n-1)^{\overline{k}} \left\langle \begin{matrix} a \\ c \end{matrix} \right\rangle_k \\
&= \sum_{k=0}^{n} (-1)^k (n-1)^{\overline{k}} \binom{n}{k} \left\langle \begin{matrix} a \\ c \end{matrix} \right\rangle_k
\end{aligned}
$$

since

$$
(-n)^{\overline{k}} = (-1)^{\overline{k}}\, k! \binom{n}{k}
$$

from the definition of the rising factorial (1.3). On the other side of the equation we have

$$
\begin{aligned}
\frac{(c-a)^{\overline{n}} (c-b)^{\overline{n}}}{c^{\overline{n}} (c-a-b)^{\overline{n}}} &= \frac{(c-a)^{\overline{n}} (c-n+1)^{\overline{n}}}{c^{\overline{n}} (c-a-n+1)^{\overline{n}}} \\
&= \frac{\left\langle \begin{matrix} 2c-a-n+1 \\ c \end{matrix} \right\rangle_n}{\left\langle \begin{matrix} 2c-a-n+1 \\ c-a \end{matrix} \right\rangle_n}
\end{aligned}
$$

if the factor $(2c-a-n+1)^{\overline{n}} / (2c-a-n+1)^{\overline{n}}$ is used.

Carlitz proved Saalschutz' formula by induction. At this stage let us replace c by c-n, and (2.2) becomes

$$
\sum_{k=0}^{n} (-1)^k (n-1)^{\overline{k}} \binom{n}{k} \left\langle \begin{matrix} a \\ c-n \end{matrix} \right\rangle_k = \frac{(n-c)^{\overline{n}} (a-c+1)^{\overline{n}}}{(a+n-c)^{\overline{n}} (1-c)^{\overline{n}}}. \tag{2.3}
$$

From this we get

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{(n-c)^{\overline{n}} (a-c+1)^{\overline{n}}}{n! (a+n-c)^{\overline{n}}} x^n &= \sum_{k=0}^{\infty} \frac{(1-c)^{\overline{n}}}{n!} x^n \sum_{k=0}^{n} (-1)^k (n-1)^{\overline{k}} \binom{n}{k} \left\langle \begin{matrix} a \\ c-n \end{matrix} \right\rangle_k \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a^{\overline{k}} (n-1)^{\overline{k}}}{k! (a+n-c)^{\overline{k}}} x^k \, \frac{(1-c)^{\overline{n-k}}}{(n-k)!} x^{n-k}
\end{aligned}
$$

in which we have used

$$
\begin{aligned}
\frac{(1-c)^{\overline{n}}}{(c-n)^{\overline{k}}} &= \frac{(1-c)(2-c)\ldots(n-c)}{(c-n)(c-n+1)\ldots(c-n+k-1)} \\
&= (-1)^k (1-c)(2-c)\ldots(n-k-c) \\
&= (-1)^k (1-c)^{\overline{n-k}}.
\end{aligned}
$$

The double sum then equals

$$\sum_{k=0}^{\infty}\frac{a^{\overline{k}}(n-1)^{\overline{k}}}{k!(a+n-c)^{\overline{k}}}x^k\sum_{n=0}^{\infty}\frac{(1-c)^{\overline{n}}}{n!}x^n \quad =\sum_{k=0}^{\infty}\frac{a^{\overline{k}}(n-1)^{\overline{k}}}{k!(a+n-c)^{\overline{k}}}x^k(1-x)^{c-1}.$$

Thus,

$$F(a,n-1:a+n-c;x) \quad =(1-x)^{1-c}F(n-c,a-c+1;a+n-c;x), \qquad (2.4)$$

where F denotes the hypergeometric function. It is customary to prove this by making use of the differential equation of the second order satisfied by $F(a,b;c;x)$. Some results for these type 2 rising binomial coefficients now follow. The first three follow easily from their respective definitions.

3. Rising binomial results

$$F(a;b,a-b;x) \quad =\sum_{r=0}^{\infty}\left\langle{a \atop b}\right\rangle_r\frac{x^r}{r!}. \qquad (3.1)$$

$$\left\langle{-n \atop -k}\right\rangle_a \quad =(-1)^a\binom{n}{k}\Big/\binom{n-a}{k}\binom{k}{k-a}a!, \qquad (3.2)$$

$$\begin{aligned}\frac{n+r}{n} &= \frac{(n+1)^{\overline{r}}}{n^{\overline{r}}}\\ &= r!\left\langle{n+1 \atop 1}\right\rangle_r. \qquad (3.3)\end{aligned}$$

Theorem:

$$\left\langle{n-1 \atop k}\right\rangle_a+\left\langle{n-1 \atop k-1}\right\rangle_a \quad =\left\langle{n \atop k}\right\rangle_a\frac{(n-1)\big(2\big(nk-k^2-a-n+1\big)=an\big)}{(n+a-1)(k-1)(n-k-1)}. \qquad (3.4)$$

$$\begin{aligned}&=\frac{(n-1)n(n+1)...(n+a-2)}{k(k+1)...(k+a-1)(n-k-1)(n-k)...(n-k+a-2)}\\ &+\frac{(n-1)n(n+1)...(n+a-2)}{(k-1)k(k+1)...(k+a-2)(n-k)(n-k+1)...(n-k+a-1)}\\ &=\frac{(n-1)n(n+1)...(n+a-2)}{k(k+1)...(k+a-2)(n-k)(n-k+1)...(n-k+a-2)}\\ &+\frac{(k+a-1)(n-k-1)+(k-1)(n-k+a-1)}{(k-1)(k+a-1)(n-k-1)(n-k+a-1)}\\ &=\frac{(n-1)(2nk-2k^2-2a-2n+2+an)}{(n+a-1)(k-1)(n-k-1)}\left\langle{n \atop k}\right\rangle_a.\end{aligned}$$

■

Another fairly simple result is

$$\left\langle \begin{matrix} -n \\ -k \end{matrix} \right\rangle_a = \frac{(-1)^a}{(n-k-a+1)^{\overline{a}}} \Big/ \binom{n-a}{k-a}. \tag{3.5}$$

Proof:

$$\begin{aligned}
\left\langle \begin{matrix} -n \\ -k \end{matrix} \right\rangle_a &= (-n)^{\overline{a}} / (-k)^{\overline{a}} (-n+k)^{\overline{a}} \\
&= \frac{(-1)^a n!(k-a)!(n-k-a)!}{k!(n-k)!(n-a)!} \\
&= (-1)^a \binom{n}{k} \Big/ \frac{(n-a)!}{(k-a)!(n-k-a)!}
\end{aligned}$$

and the result follows since

$$\begin{aligned}
\frac{(n-k-a)!}{(n-k)!} &= \frac{1}{(n-k)(n-k-1)...(n-k-a+1)} \\
&= \frac{1}{(n-k-a+1)^{\overline{a}}}.
\end{aligned}$$

4. Concluding Comments

Since $\left\langle \begin{matrix} i \\ j \end{matrix} \right\rangle_1 = \frac{i}{j(i-j)}$, this is not a straightforward generalization of the ordinary binomial coefficient. However, Jordan [3, pp.70 ff.] dealt with generalized binomial coefficients of the form

$$\binom{x}{n}_h = \frac{x(x-h)(x-2h)...(x-nh+h)}{n!} \tag{4.1}$$

so that $\binom{x}{n}_1 = \binom{x}{n}$. It would be of interest to extend these to rising factorial Jordan coefficients defined by

$$\left\{ \begin{matrix} x \\ n \end{matrix} \right\}_{ha} = \frac{a(a+h)(a+2h)...(a+xh-h)}{a(a+h)(a+2h)...(a+xh-nh-h)}, \tag{4.2}$$

for example, and to investigate their properties.

Carlitz' note on a theorem of Glaisher [1] may provide a means to investigate congruences for rising factorials. For

$$x^{\overline{p}} = x^{p-1} + A_1 x^{p-2} + ... + A_{p-1},$$

Glaisher proved that

$$\tfrac{1}{2}p(p-r)A_{r-1} - A_r \equiv 0 \pmod{p^4}. \tag{4.3}$$

Carlitz extended this to

$$\tfrac{1}{2}p(p-r)A_{r-1} - A_r \equiv r(r-1)(r-2)B_{r-3}p^4/24(r-3) \pmod{p^5} \tag{4.4}$$

where B_r is a Bernoulli number.

References

1. Carlitz, L. Note on a Theorem of Glaisher. *Journal of the London Mathematical Society*. 28 (1953): 245-246.
2. Carlitz, L. Generating Functions. *The Fibonacci Quarterly*. Vol.7 (1969), pp.359-393.
3. Jordan, C. *Calculus of Finite Differences*. New York: Chelsea.
4. Macmahon, P.A. *Combinatory Analysis.* Cambridge: Cambridge University Press, 1916.
5. Riordan, J. *An Introduction to Combinatorial Analysis*. New York: Wiley, 1958, p.3.
6. Shannon, A.G. Some *q*-Binomial Coefficients Formed from Rising Factorials. *Notes on Number Theory and Discrete Mathematics.* 12 (2006): 13-20.
7. Wright, E.M. An Identity and Applications. *American Mathematical Monthly*. Vol.75 (1968), pp.711-714.