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Note on φ, ψ and σ functions

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In the present remark we shall formulate and discuss some extremal problems, related to arithmetic functions φ, σ and ψ (see, e.g., [1]).

Let

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

be the prime number factorization of n, then

$$\varphi(n) = n.\prod_{i=1}^{k}(1 - \frac{1}{p_i});$$

$$\psi(n) = n.\prod_{i=1}^{k}(1 + \frac{1}{p_i});$$

$$\sigma(n) = \prod_{i=1}^{k}\frac{p_i^{\alpha_i+1} - 1}{p_i - 1}.$$

Let $\underline{max}(n)$ be the maximal prime divisor of n. We shall prove the following

Theorem 1. For each natural number n

$$\max_{d/n}\varphi(d)\sigma(\frac{n}{d}) = \varphi(\underline{max}(n)).\sigma(\frac{n}{\underline{max}(n)}). \tag{1}$$

Proof. First, we shall prove that

$$\varphi(p)\sigma(p^{a-1}m) \geq \varphi(p^s)\sigma(p^{a-s}m), \tag{2}$$

where p is a prime number, m, s and $a \geq 2$ are natural numbers, $(m,p) = 1$ and $1 \leq s \leq a-1$. Sequentially we obtain:

$$\varphi(p)\sigma(p^{a-1}m) - \varphi(p^s)\sigma(p^{a-s}m) = ((p-1).\frac{p^a - 1}{p-1} - p^{s-1}.(p-1).\frac{p^{a-s+1} - 1}{p-1}).\sigma(m)$$

$$= (p^a - 1 - p^{s-1}.(p^{a-s+1} - 1)).\sigma(m) = (p^{s-1} - 1).\sigma(m) \geq 0.$$

Let the prime numbers p and q satsfy: $p > q$, $p \geq 5$ and let $n = p^a.q^b.m$, where $a, b, m \geq 1$ are natural numbers. Then $p \geq q + 2$ and for arbitrary s so that $1 \leq s \leq b$ we obtain from (2):

$$\varphi(p)\sigma(p^{a-1}q^bm) - \varphi(q^s)\sigma(p^aq^{b-s}m) \geq \varphi(p)\sigma(p^{a-1}q^bm) - \varphi(q)\sigma(p^aq^{b-s}m)$$

$$= \sigma(m).((p-1)\frac{p^a-1}{p-1}\frac{q^{b+1}-1}{q-1} - (q-1)\frac{p^{a+1}-1}{p-1}\frac{q^b-1}{q-1})$$

$$= \frac{\sigma(m)}{(p-1)(q-1)}.((p^{a+1}-p-p^a+1)(q^{b+1}-1) - (p^{a+1}-1).(q^{b+1}-q^b-q+1))$$

$$= \frac{\sigma(m)}{(p-1)(q-1)}.(-2p^{a+1}-pq^{b+1}+p-p^aq^{b+1}+p^a+2q^{b+1}p^{a+1}q-q+p^{a+1}q^b-q^b-p^{a+1})$$

$$= \frac{\sigma(m)}{(p-1)(q-1)}.(p^aq^b(p-q)+p^a(pq-2p+1)-q^b(pq-2q+1)+p-q)$$

$$> \frac{\sigma(m)}{(p-1)(q-1)}.(2p^aq^b - q^b(pq-2q+1))$$

$$= \frac{\sigma(m)}{(p-1)(q-1)}.q^b.(2p^a-pq+2q-1) > 0.$$

Finally, let $p = 3$, $q = 2$ and let $n = 3^a.2^b.m$, where $a, b, m \geq 1$ are natural numbers. Then for arbitrary s so that $1 \leq s \leq b$ we obtain from (2):

$$\varphi(3)\sigma(2^b3^{a-1}m) - \varphi(2^s)\sigma(2^{b-s}3^am) \geq \varphi(3)\sigma(2^b3^{a-1}m) - \varphi(2)\sigma(2^{b-1}3^am)$$

$$= \sigma(m).((2^{b+1}-1)(3^a-1) - (2^b-1).\frac{3^{a+1}-1}{2})$$

$$= \frac{\sigma(m)}{2}.((2^{b+2}-2)(3^a-1) - (2^b-1).(3^{a+1}-1))$$

$$= \frac{\sigma(m)}{2}.(2^{b+2}3^a - 2^{b+2} - 2.3^a + 2 - 2^b.3^{a+1} + 2^b + 3^{a+1} - 1)$$

$$= \frac{\sigma(m)}{2}.(2^b3^a - 3.2^b + 3^a + 1) > \frac{\sigma(m)}{2}.(2^b3^a - 3.2^b) \geq 0.$$

Therefore, (1) is proved. Similarly is proved

Theorem 2. For each natural number n

$$\max_{d/n} \varphi(d)\psi(\frac{n}{d}) = \varphi(\underline{max}(n)).\psi(\frac{n}{\underline{max}(n)}).$$

Finally, we must note that equalities from the forms (1) and (3) are not valid for ψ and σ functions, because:

$$\psi(3)\sigma(6) - \psi(2)\sigma(9) = 4.3.4 - 3.13 = 48 - 397 > 0$$

while

$$\psi(3)\sigma(12) - \psi(2)\sigma(18) = 4.7.4 - 3.13.3 = 112 - 117 < 0.$$

Reference:

[1] Nagell T., Introduction to number theory, John Wiley & Sons, New York, 1950.