On Steiner Loops of cardinality 20

By

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Abstract: It is well known that there are five classes of sloops of cardinality 16 "SL(16)s" according to the number of sub-SL(8)s [4, 6]. In this article, we will show that there are exactly 8 classes of nonsimple sloops and 6 classes of simple sloops of cardinality 20 "SL(20)s". Based on the cardinality and the number of (normal) subsloops of SL(20), we will construct in section 3 all possible classes of nonsimple SL(20)s and in section 4 all possible classes of simple SL(20)s. We exhibit the algebraic and combinatoric properties of SL(20)s to distinguish each class.

So we may say that there are six classes of SL(20)s having one sub-SL(10) and n sub-SL(8)s for n = 0, 1, 2, 3, 4 or 6. All these sloops are subdirectly irreducible having exactly one proper homomorphic image isomorphic to SL(2). For n = 0, the associated SL(20) is a nonsimple subdirectly irreducible having one sub-SL(10) and no sub-SL(8)s. Indeed, the associated Steiner quasigroup SQ(19) of this case supplies us with a new example for a semi-planar SQ(19), where the smallest well-known example of semi-planar squags is of cardinality 21 "cf. [3]".

It is well known that there is a class of planar Steiner triple systems (STS(19)s) due to Doyen [7], where the associated planar SL(20) has no sub-SL(10) and no sub-SL(8). In section 4 we will show that there are other 6 classes of simple SL(20)s having n sub-SL(8)s for n = 0, 1, 2, 3, 4, 6, but no sub-SL(10)s. It is well-known that a sub-SL(m) of an SL(2m) is normal. In the last theorem of this section, we give a necessary and sufficient condition for a sub-SL(2) to be normal of an SL(2m). Accordingly, we have shown that if a sloop SL(20) has a sub-SL(10) and 12 sub-SL(8), then this sloop is isomorphic to the direct product SL(10) × SL(2) and if a sloop SL(20) has 12 sub-SL(8)s and no sub-SL(10), then this sloop is a subdirectly irreducible having exactly one proper homomorphic image isomorphic to SL(10). In section 5, we describe how can one construct an example for each class of simple and of nonsimple SL(20)s.

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Introduction:

A Steiner loop (briefly sloop) is a groupoid $S = (S; \cdot, 1)$ with neutral element 1 satisfying the identities:

$$x \cdot x = 1, \quad x \cdot y = y \cdot x, \quad x \cdot (x \cdot y) = y.$$

A Steiner quasigroup (briefly squag) is a groupoid $Q = (Q; *)$ satisfying the identities:
\[ x \cdot x = x, \quad x \cdot y = y \cdot x, \quad x \cdot (x \cdot y) = y. \]

Notice that both squags and sloops are quasigroups [5, 8].

We use the abbreviations \( \text{SL}(n) \) and \( \text{SQ}(n) \) for a sloop and a squag of cardinality \( n \), respectively. A sloop is called Boolean (or Boolean group) if it satisfies the associative law \((x \cdot y) \cdot z = x \cdot (y \cdot z)\).

A Steiner triple system is a pair \((P; B)\), where \( P \) is a set of points and \( B \) is a set of 3-element subsets of \( P \) called blocks such that for distinct points \( p_1, p_2 \in P \), there is a unique block \( b \in B \) with \( \{p_1, p_2\} \subseteq b \). If the cardinality of the set of points \( P \) is equal to \( n \), the Steiner triple system \((P; B)\) will be denoted by \( \text{STS}(n) \). It is well known that a necessary and sufficient condition for the existence of an \( \text{STS}(n) \) is \( n \equiv 1 \) or 3 \((\text{mod} \ 6)\) [8, 11]. There is a one to one correspondence between sloops (squags) and Steiner triple systems given by the relation:

\[ x \cdot y = z \iff \{x, y, z\} \text{ is a block} \ [8, 11, 12]. \]

Quackenbush [12] proved that the congruences of sloops are permutable, regular, and Lagrangian. A subsloop \( S \) of a sloop \( L \) is called normal iff \( (x \cdot y) \cdot S = x \cdot (y \cdot S) \) for all \( x, y \in L \). Also in [12] was proved that if \( S \) is a subsloop of \( L \) and \(|L| = 2 |S|\), then \( S \) is normal.

A three distinct points \( x, y, z \) are called a triangle if \( \{x, y, z\} \) does not form a block (or equivalently, if \( \{x, y, z\} \) does not contains the identity element and \( x \cdot y \neq z \)). An \( \text{STS} \) is planar if it is generated by every triangle \( . \) A planar \( \text{STS}(n) \) exists for each \( n \geq 7 \) and \( n \equiv 1 \) or 3 \((\text{mod} \ 6)\) [7]). The associated squag and sloop of a planar triple system are also called planar. Quackenbush [12] has shown that the only nonsimple finite planar sloop (squag) has 8 (9) elements. Accordingly, we may say that there is always a simple \( \text{SQ}(n) \) and a simple \( \text{SL}(n +1) \) for all \( n > 9 \) and \( n \equiv 1 \) or 3 \((\text{mod} \ 6)\).

A semi-planar sloop (squag) is a simple sloop (squag) each of whose triangles generates either the whole sloop (squag) or a sub-\( \text{SL}(8) \) (a sub-\( \text{SQ}(9) \)). The associated \( \text{STS} \) with a semi-planar sloop (squag) is said to be a semi-planar \( \text{STS} \) or more precisely a semi-7-planar \( \text{STS} \) (a semi-9-planar \( \text{STS} \)), if each of whose triangles generates either the whole \( \text{STS} \) or a sub-\( \text{STS}(7) \) (a sub-\( \text{STS}(9) \)) [2, 3, 12].

The author in [2, 3] has given a construction of semi-planar sloops of cardinality \( 2(n +1) \) and a construction of semi-planar squags of cardinality \( 3n \) for each possible \( n > 3 \).

An extensive study of sloops can be found in [5, 8, 11]. We will use in this article some basic concepts of universal algebra [9] and some other concepts of graph theory [10].

There is a well-known classification of all \( \text{SL}(16) \)s based on the number of sub-\( \text{SL}(8) \)s. In fact, there are four classes from five classes of \( \text{SL}(16) \)s are subdirectly irreducible [4, 6, 11]. The next admissible order for sloops is of cardinality 20. In this article we are not interesting
in counting the number of distinct \( \text{SL}(20) \)s, but we exhibit the algebraic and combinatoric properties of \( \text{SL}(20) \)s to distinguish each class.

We may divide all \( \text{SL}(20) \)s based on the cardinality and the number of the (normal) subsloops of an \( \text{SL}(20) \) into the following classes. Note that a sub-\( \text{SL}(10) \) in an \( \text{SL}(20) \) is always normal.

1- There are planar \( \text{STS}(19) \)s due to Doyen [7]. The associated planar sloops \( \text{SL}(20) \)s and the associated planar squags \( \text{SQ}(19) \)s are simple. Indeed, these planar \( \text{SL}(20) \)s (or planar \( \text{SQ}(19) \)s) have no nontrivial subsloops (subsquags).

2- In section 3, we will show that there are 7 classes of \( \text{SL}(20) \)s. Each of these classes has exactly one sub-\( \text{SL}(10) \) and \( n \) sub-\( \text{SL}(8) \)s (for \( n = 0, 1, 2, 3, 4, 6, 12 \)). For \( n = 12 \), we have a class of \( \text{SL}(20) \)s containing one sub-\( \text{SL}(10) \) and 12 sub-\( \text{SL}(8) \)s, we will show that this sloop must be isomorphic to the direct product \( \text{SL}(10) \times \text{SL}(2) \). Also, all \( \text{SL}(20) \)s of the other six classes (for \( n = 0, 1, 2, 3, 4, 6 \)) are nonsimple subdirectly irreducible. For \( n = 0 \), the associated \( \text{SL}(20) \) is nonsimple subdirectly irreducible having one sub-\( \text{SL}(10) \) and no sub-\( \text{SL}(8) \)s. Indeed, the associated \( \text{SQ}(19) \) with this case supplies us with a new example for a semi-planar \( \text{SQ}(19) \) (or a semi-9-planar \( \text{STS}(19) \)). Of course, this class of semi-planar \( \text{SQ}(19) \)s are not planar and have exactly one sub-\( \text{SQ}(9) \), but no sub-\( \text{SQ}(7) \). It will be convenient to note at this point that the smallest well-known example of semi-planar squags is of cardinality 21 " cf. [3]". An example of this new case (a semi-planar \( \text{SQ}(19) \)) will be given in section 5.

3- In section 4, we will show that there are five classes of semi-planar \( \text{SL}(20) \)s based on the number \( n = 1, 2, 3, 4 \) or 6 of sub-\( \text{SL}(8) \)s. All of these semi-planar \( \text{SL}(20) \)s are simple and not planar. In addition, the associated \( \text{SQ}(19) \)s of these classes are simple and each triangle generates a sub-\( \text{SQ}(7) \) or the whole \( \text{SQ}(19) \).  

4- According to the construction given in section 4, there is a class of \( \text{SL}(20) \)s having no sub-\( \text{SL}(10) \) and 12 sub-\( \text{SL}(8) \)s. We will show that these sloops are subdirectly irreducible and have exactly one proper congruence with classes of cardinality two (one proper homomorphic image isomorphic to \( \text{SL}(10) \)).

In fact, these are all classes of \( \text{SL}(20) \)s. In section 5 we describe how can one construct an example for each class.

2. Construction of an \( \text{SL}(2n) = 2 \otimes \alpha L_1 \).

Using the doubling construction \( \text{SL}(2n) \)s [11], we will study in this section some properties of subsloops of \( \text{SL}(2n) \)s.
Let $T_1 = (P^*; B_1)$ be an $STS(n)$ and its corresponding sloop $L_1 = (P_1; \cdot, e)$, where $P^*_1 = \{a_1, \ldots, a_n\}$ and $P_1 = P^*_1 \cup \{e\}$. Consider the set of 1-factors defined by $F_i = \{a_i a_k : a_i \cdot a_k = a_i \text{ and } a_i, a_k, a_1 \in P_1\}$, then the class $F = \{F_1, F_2, \ldots, F_n\}$ forms a 1-factorization of the complete graph $K_n$ on the set of vertices $P_1$.

By taking the set $P_2 = \{b_1, b_2, \ldots, b_n\}$ with $P_1 \cap P_2 = \emptyset$ and $G_i = \{b_i b_k : a_i \cdot a_k = a_i \text{ for } i \not\in \{l, k\}\}$, then the class of 1-factors $G = \{G_1, G_2, \ldots, G_n\}$ forms a 1-factorization of the complete graph $K_n$ on the set of vertices $P_2$. There is a well-known construction of an $STS(2n+1) = (P^*; B)$ [11], where $P^* = P^*_1 \cup P_2$ and the set of triples $B = B_1 \cup \{b_i b_k a_i : b_i b_k \in G_{a(i)}\}$ for any permutation $a$ on the set $\{1, \ldots, n\}$. The constructed $STS(2n+1) = (P^*; B)$ and the associated sloop $SL(2n+2) = (P; \cdot, e)$ will be denoted by $2 \otimes_a T_1$ and $2 \otimes_a L_1$, respectively.

If we choose the permutation $a = \text{the identity}$, then the constructed sloop $L = 2 \otimes_a L_1$ isomorphic to the direct product of $SL(n+1) = L_1$ and the 2-element sloop $SL(2)$. We observe that $L_1$ is a normal subsloop of $2 \otimes_a L_1$ for any permutation $a$.

It is easy to prove the following fact.

**Lemma 1.** Let $2 \otimes_a L_1 = (P = P_1 \cup P_2; \cdot, e)$ be the constructed sloop of cardinality $2n$ with the subsloop $L_1 = (P_1; \cdot, e)$ of cardinality $n$. Then any subsloop $S$ of $L$ with $S \cap P_1 \neq \emptyset$ satisfies $|S \cap P_1| = (1/2)|S| = 1, 2 \text{ or } 4$.

In the following we consider the $SL(10) = L_1 = (P_1 = P^*_1 \cup \{e\}; \cdot, e)$ associated with the $STS(9) = (P^*_1; B_1)$, where $P^*_1 = \{a_1, \ldots, a_9\}$. Also, we consider the set $P_2 = \{b, b_1, b_2, \ldots, b_9\}$ with $P_1 \cap P_2 = \emptyset$ and $a$ be any permutation on the set $N = \{1, 2, \ldots, 9\}$ to construct the $SL(20) = 2 \otimes_a L_1$. Moreover, we consider the $STS(9) = (N; X)$, where $X$ is defined by: $\{i, j, k\} \in X$ if and only if $\{a_i, a_j, a_k\} \in B_1$.

The constructed $STS(19)$ is given by $2 \otimes_a T_1 = (P^* = P^*_1 \cup P_2; B = B_1 \cup B_{12})$, where $B_{12} = \{a_i b_j b_k : b_j b_k \in G_{a(i)}\}$. Let $2 \otimes_a L_1 = (P; \cdot, e)$ be the associated sloop with $STS(19) = 2 \otimes_a T_1$.

For each block $\{a_i, a_j, a_k\} \in B_1$ there is a sub-1-factorization $f = \{f_i = \{e a_i a_j a_k\}, f_j = \{e a_j a_i a_k\}, f_k = \{e a_k a_i a_j\}\}$ of $F$. Conversely, if there is a sub-1-factorization on the 4-element subset $\{e, a_i, a_j, a_k\}$, then $\{a_i, a_j, a_k\}$ is a block in $B_1$. This means that there is a one-one correspondence between the set of blocks of $B_1$ and the sub-1-factorizations of $K_4$ in $F$.

**Lemma 2.** Each of the 1-factorization $F$ on the set $P_1$ and the 1-factorization $G$ on the set $P_2$ has exactly 12 sub-1-factorizations of $K_4$. 
Theorem 5. The constructed sloop $2 \otimes_\alpha L_1 = (P = P_1 \cup P_2; \cdot, e)$ is isomorphic to the direct product of the subsloop $\text{SL}(10) = L_1$ and the 2-element sloop $\text{SL}(2)$, if and only if 2 $\otimes_\alpha L_1$ has 12 sub-$\text{SL}(8)s$, otherwise $2 \otimes_\alpha L_1$ is nonsimple subdirectly irreducible. Moreover, the constructed sloop $2 \otimes_\alpha L_1$ has exactly n subloops of cardinality 8 if and only if the
permutation $\alpha$ transfers $n$ lines into $n$ lines of $X$ for $n = 0, 1, 2, 3, 4, 6, 12$. Where $X$ is the set of lines of the affine plane over $GF(3)$.

**Proof.** Let $2 \otimes_{\alpha} L_1$ have 12 sub-$SL(8)$s, then $\alpha(X) := \{\{\alpha(i), \alpha(j), \alpha(k)\} : \text{for all } \{i, j, k\} \in X\} = X$. Consider the map $\varphi$ from $2 \otimes_{\alpha} L_1$ to the direct product $L_1 \times \{0, 1\}$ by $\varphi(e) = (1, 0)$, $\varphi(b) = (1, 1)$, $\varphi(a_i) = (a_i, 0)$ and $\varphi(b_i) = (a_i^{-1}(i), 1)$. It is routine matter to proof that $\varphi$ is an isomorphism. Notice that $\varphi(a_i b_j) = \varphi(b_k)$ if $b_j b_k \in G_{a(i)}$, so $\{\alpha(i), j, k\}$ is a line in $X$. Also, $\varphi(b_k) = (a_i^{-1}(k), 1)$ and $\varphi(a_i) \varphi(b_i) = (a_i, 0) (a_i^{-1}(i), 1) = (a_i, a_i^{-1}(i), 1)$, but $\alpha^{-1}\{\alpha(i), j, k\} = \{i, \alpha^{-1}(j), \alpha^{-1}(k)\}$ is also a line in $X$, so $\varphi(a_i) \varphi(b_j) = \varphi(b_k)$.

Since $2 \otimes_{\alpha} L_1$ has exactly one normal subloop $L_1$ of cardinality 10, so $2 \otimes_{\alpha} L_1$ is not simple. Another possible normal subloop is the 2-element subloop $C_2$ with $C_2 \cap L_1 = \{e\}$. But if $2 \otimes_{\alpha} L_1$ contains $C_2$ as a normal subloop, then $2 \otimes_{\alpha} L_1$ is isomorphic to the direct product $SL(10) \times SL(2)$ and has exactly 12 sub-$SL(8)$s. Therefore, if $2 \otimes_{\alpha} L_1$ has $n$ sub-$SL(8)$s with $n < 12$, then the congruence lattice of $2 \otimes_{\alpha} L_1$ has only the normal subloop $L_1$. Hence $2 \otimes_{\alpha} L_1$ is subdirectly irreducible for all possible $n < 12$.

Let $\alpha$ transfer the line $\{i, j, k\} \in X$ into the line $\{\alpha(i), \alpha(j), \alpha(k)\} \in X$. According to Lemmas 3 and 4, we may directly say that $S = \{e, a_i, a_j, a_k, b, b_{a(i)}, b_{a(j)}, b_{a(k)}\}$ forms a subloop. Since $\alpha$ is a permutation on the set of points $N = \{1, 2, \ldots, 9\}$ of the affine plane over $GF(3)$, the possible values of the number $n$ of lines of $X$ transferred into lines are $0, 1, 2, 3, 4, 6$ or 12. This completes the proof of the theorem.

In fact, there is another class of subdirectly irreducible $SL(20)$s having exactly one proper normal sub-$SL(2)$ but no sub-$SL(10)$. It will be described in the next section.

In [1] the author has given a construction of subdirectly irreducible sloops of cardinality $2m$. This construction supplies us with an example of a subdirectly irreducible $SL(20)$ of one of these classes.

According to the results given in [12], the variety $V_1$ generated by the $SL(10) = L_1$ covers the smallest nontrivial subvariety $V_0$ (the class of all Boolean sloops ). The constructed subdirectly irreducible sloop $2 \otimes_{\alpha} L_1 = SL(20)$ contains always a subloop of cardinality 10 and has only one proper homomorphic image isomorphic to the Boolean sloop $SL(2)$. According to the result given [1], we may deduce that each of these sloops $SL(20) = 2 \otimes_{\alpha} L_1$ generates a variety $V_2$ covers the variety $V_1$.

4. Semi-planar sloops of cardinality 20

A semi-planar sloop is a simple sloop each of whose triangles generates either the whole sloop or a sub-$SL(8)$ " cf. [2, 15]." The associated STSs with the semi-planar sloops will also be called semi-planar (or more precisely semi-7-planar). Each semi-planar $SL(20)$
contains sub-SL(8)s but no sub-SL(10)s. Based on the number \( n \) of sub-SL(8)s of SL(20), we will determine all classes of simple SL(20)s. So we have six distinct classes of simple sloops, one of them is the class of planar SL(20) and the other five classes are semi-planar SL(20)s. In [2] the author has given another construction of a semi-planar sloop SL(2m). This construction supplies us with exactly one class among these five classes.

We will modify the construction of the subdirectly irreducible SL(20) = \( 2 \otimes_a L_1 = (P = P_1 \cup P_2; \cdot, e) \) to a construction of semi-planar sloop denoted by \( 2 \otimes_a L_1 \). Let the associated STS(19) of the constructed subdirectly irreducible SL(20) = \( 2 \otimes_a L_1 \) has a sub-STS(7) on the set of elements \( A^* = \{a_i, a_j, a_k, b, b_{a(i)}, b_{a(j)}, b_{a(k)}\} \); i.e., \( \{i, j, k\} \) and \( \{a(i), a(j), a(k)\} \) are lines in \( X \). We will interchange the set of blocks:

\[ \mathcal{H} = \{\{a_i, a_j, a_k\}, \{a_i, b_{a(i)}, b_{a(k)}\}, \{a_j, b_{a(j)}, b_{a(k)}\}, \{a_k, b_{a(k)}, b_{a(j)}\}\} \]

with the set of triples

\[ R = \{\{b_{a(i)}, b_{a(j)}, b_{a(k)}\}, \{b_{a(i)}, a_j, a_k\}, \{b_{a(j)}, a_i, a_k\}, \{b_{a(k)}, a_i, a_j\}\} \]

to get again an STS(19) = \( (P^* = P_1^* \cup P_2^*; \cdot, e) \) denoted by \( 2 \otimes_a T_1 \). The associated sloop will be denoted by \( 2 \otimes_a L_1 = (P = P_1 \cup P_2; \cdot, e) \). Notice that the difference between the binary operations “\( \cdot \)” and “\( \cdot, e) \) is only restricted on the subset of elements of \( A^* \); i.e., \( x \cdot y = x \cdot y \) for all \( x, y \in P - A^* \).

The next lemma is one of the main results of this section. To show that the new construction \( 2 \otimes_a L_1 \) is a semi-planar sloop such that \( \alpha \) transfers at least one line into a line and at most 6 lines into 6 lines of the affine plane over GF(3).

**Theorem 6.** The constructed sloop \( 2 \otimes_a L_1 = (P = P_1 \cup P_2; \cdot, e) \) has no sub-SL(10). Also, \( 2 \otimes_a L_1 \) is a semi-planar sloop having \( n \) sub-SL(8)s for each \( n = 1, 2, 3, 4 \) or 6. Where \( n \) is the number of lines of the affine plane over GF(3) transferred into lines by the permutation \( \alpha \).

**Proof.** Let \( S = \{x, y, z\} \) be a triangle in \( 2 \otimes_a L_1 \). At first, we want to prove that the subloop \( \langle S \rangle \) in \( 2 \otimes_a L_1 \) is equal to the whole sloop \( 2 \otimes_a L_1 \) or a sub-SL(8).

Assume that \( \langle S \rangle \cap A \leq 2 \), where \( A = \{e, a_i, a_j, a_k, b, b_{a(i)}, b_{a(j)}, b_{a(k)}\} \), then the subloop \( \langle S \rangle \) in the sloop \( 2 \otimes_a L_1 \) is the same as the subloop \( \langle S \rangle \) in \( 2 \otimes_a L_1 \). So if \( \langle S \rangle \) is a sub-SL(10), then \( \langle S \rangle = \langle S \rangle = L_1 \) contradicting the fact that \( a_i \cdot a_j = b_{a(k)} \) in \( 2 \otimes_a L_1 \).

Also, if \( \langle S \rangle \cap A = \{e, b, a_i, b_{a(i)}\}, \{e, b, a_j, b_{a(j)}\} \) or \( \{e, b, a_k, b_{a(k)}\} \), then the subloop \( \langle S \rangle \) in the sloop \( 2 \otimes_a L_1 \) is the same as the subloop \( \langle S \rangle \) in \( 2 \otimes_a L_1 \). For the same reason, if \( \langle S \rangle \) is a sub-SL(10), then \( \langle S \rangle = \langle S \rangle = L_1 \) contradicting the fact that \( b \in \langle S \rangle \).

Moreover if \( \langle S \rangle \cap A \geq 4 \), then \( \langle S \rangle = A \); i.e., \( \langle S \rangle \) is a sub-SL(8).

Now Assume that \( \langle S \rangle = 10 \) and \( \langle S \rangle \cap A = \{e, b_{a(i)}, a_j, a_k\}, \{e, b_{a(j)}, a_i, a_k\}, \{e, b_{a(k)}, a_i, a_j\} \) or \( \{e, b_{a(i)}, b_{a(j)}, b_{a(k)}\} \).
Each of these four blocks contains at least one element $b_{a(i)}$ lying in $P_2$, for $t = i$, $j$ or $k$. If $b_r$ or $a_s \in \langle S \rangle - A$, then $b_{a(i)} \not\in P_1$ and $b_{a(i)} \not\in P_2$. This means that the 6-element subset \( \langle S \rangle - A \) consists of two 3-element subsets \( \{a_{s1}, a_{s2}, a_{s3}\} \subseteq P_1 \) and \( \{b_{r1}, b_{r2}, b_{r3}\} \subseteq P_2 \). For the three case \( \langle S \rangle \cap A = \{e, b_{a(i)}, a_r, a_k\} \) or $\{e, b_{a(k)}, a_r, a_i\}$, we have $a_i = \{a_{s1}, a_{s2}, a_{s3}\}$ \( \not\in \{a_{s1}, a_{s2}, a_{s3}\} \) and $a_r = \{a_{s1}, a_{s2}, a_{s3}\} \cap \{e, a_r, a_k\} = \emptyset$ for $t = i, j$ or $k$, this means that \( \langle S \rangle \) consists of a 4-element subset of $P_2$ and more than 6 elements lying in $L_1$, hence \( \langle S \rangle \) has more than 10 elements, so \( \langle S \rangle \) must be equal to \( 2 \otimes_\alpha L_1 \).

For the case \( \langle S \rangle \cap A = \{e, b_{a(i)}, b_{a(j)}, b_{a(k)}\} \), the set \( \langle S \rangle - A \) contains \( \{a_{s1}, a_{s2}, a_{s3}\} \subseteq P_1 \) and \( \{b_{r1}, b_{r2}, b_{r3}\} \subseteq P_2 \). Let \( \langle S \rangle^* \) be the associated STS(9) with \( \langle S \rangle \). Since \( \{a_{s1}, a_{s2}, a_{s3}\} \cap \{a_i, a_j, a_k\} = \emptyset \), the triple \( \{a_{s1}, a_{s2}, a_{s3}\} \) forms a block of \( \langle S \rangle^* \). If \( \langle S \rangle \) is a sub-SL(10), then the triple \( \{b_{r1}, b_{r2}, b_{r3}\} \) also be a block of \( \langle S \rangle^* \) contradicting the fact that the construction \( 2 \otimes_\alpha L_1 \) contains exactly one block lying completely in $P_2$ that is the block \( \{b_{a(i)}, b_{a(j)}, b_{a(k)}\} \subseteq P_2 \). Therefore, the subloop \( \langle S \rangle \) generated by any triangle $S$ is equal to a sub-SL(8) or the whole sloop \( 2 \otimes_\alpha L_1 \). This means that \( 2 \otimes_\alpha L_1 \) has no sub-SL(10)s for all $n = 1, 2, 3, 4, 6$ or 12.

Secondly, we have to prove that \( 2 \otimes_\alpha L_1 \) has no proper congruence for $n = 1, 2, 3, 4$ and 6. Assume that \( 2 \otimes_\alpha L_1 \) has a congruence $\theta$ with $[e] \theta = \{e, x\}$. If $[e] \theta \cap A = \{e\}$, then $[A] \theta$ is a sub-SL(16) which is impossible. Hence $[e] \theta \cap A = \{e\}$.

Now assume that $[e] \theta = \{e, b_{a(i)}\} \subseteq A$ and suppose that $\{a_i, b_r, b_s\}$ is a block such that $\{a_i, b_r, b_s\} \cap \{a_j, a_k, a_l\} = \emptyset$ for $i \neq j$, so we have $[e] \theta \cup [a_i] \theta \cup [a_j] \theta \cup [a_k] \theta = \{e, a_i, a_j, a_k, a_l, b_r, b_s, b_{a(i)}, b_{a(j)}, b_{a(k)}\}$, where $b_{a(i)} = b_r \neq b_{a(k)}$ contradicting that $[e] \theta \cup [a_i] \theta \cup [a_j] \theta \cup [a_k] \theta \cup [b_i] \theta$ is an 8-element subloop.

Now assume that $[e] \theta = \{e, b_{a(i)}\} \subseteq A$ and suppose that $\{a_i, b_r, b_s\}$ is a block such that $\{a_i, b_r, b_s\} \cap A = \{a_j\}$ for $i \neq j$. So we have $[e] \theta \cup [a_i] \theta \cup [b_r] \theta \cup [b_s] \theta = \{e, a_i, a_j, a_k, b_r, b_s, b_{a(i)}, b_{a(i)} \cap a_j, a_k, b_r, b_s\}$, where $b_{a(i)} = a_i = b_{a(i)} \cap a_j, a_k, b_r, b_s$.

Finally, for $n = 12$, the permutation $\alpha$ transfers each line of $X$ into a line. Then the constructed sloop $3 \otimes_\alpha L_1$ has 12 SL(8)s but no SL(10)s. It will be shown that $3 \otimes_\alpha L_1$ is not
The next theorem shows that $2 \otimes _{\alpha} L_1$ in this case is subdirectly irreducible having exactly one normal subsloop that is a sub-$SL(2)$.

**Theorem 7.** Let $L$ be a sloop of cardinality $2m$. A subsloop $S = \{e, x\}$ is normal if and only if $L$ contains $(m - 1)(2m - 4)/12$ sub-$SL(8)$s including the element $x$.

**Proof.** If $S$ is a normal subsloop of $L$, then $L/S$ is an $SL(m)$. An $SL(m)$ has $(m - 1)(m - 2)/6$ 4-element subsloops, which implies that $L$ has $(m - 1)(m - 2)/6$ sub-$SL(8)$s containing $S$.

On the other direction, the number of triangles of an $SL(2m)$ passing through a fixed point $x$ is equal to $(2m - 2)(2m - 3)/2 - (2m - 2)/2 = (m - 1)(2m - 4)$ and the number of triangles of an $SL(8)$ passing through the fixed point $x$ is equal to 12, then the maximum number of sub-$SL(8)$s of an $SL(20)$ containing $x$ is equal to $(m - 1)(2m - 4)/12$. This means that if $L$ contains $(m - 1)(2m - 4)/12$ sub-$SL(8)$s passing through $x$, then each triangle in $L$ generates a sub-$SL(8)$. Let $y, z \in L - \{e, x\}$. If $\{e, x, y, z\}$ forms a sub-$SL(4)$ then $y \cdot z \cdot (S) = y \cdot (z \cdot (S))$. If $\{e, x, y, z\}$ does not form a sub-$SL(4)$, then $\{x, y, z\}$ is a triangle in $L$ and the subsloop generated by $\{x, y, z\}$ is an $SL(8)$. It is well known that an $SL(8)$ is always Boolean. This implies that $y \cdot z \cdot (S) = y \cdot (z \cdot (S))$. So $S$ is normal. This completes the proof of the lemma.

According to the above theorem, if the constructed $SL(2m) = 2 \otimes _{\alpha} L_1$ has a simple $SL(m) = L_1$ and $(m - 1)(m - 2)/6$ sub-$SL(8)$s passing through sub-$SL(2) = \{e, x\}$, so $\{e, x\}$ is normal. Since $L_1$ is simple then $L_1 \cap \{e, x\} = \{e\}$. According to the definition of the constructed $2 \otimes _{\alpha} L_1$, we have $x = b$, so the subsloops $L_1$ and $\{e, b\}$ are normal, then $2 \otimes _{\alpha} L_1$ is isomorphic to the direct product $SL(m) \times SL(2)$. This result agrees with result of Theorem 5.

Also, for $m = 10$ in the above theorem and according to Theorem 6, we may say that for $n = 12$ the constructed sloop $SL(20) = 2 \otimes _{\alpha} L_1$ has $(10 - 1)(20 - 4)/12 = 12$ sub-$SL(8)$s passing through sub-$SL(2) = \{e, b\}$, but no sub-$SL(10)$. So $2 \otimes _{\alpha} L_1$ has exactly one proper congruence $\theta$ with $[e]\theta = \{e, b\}$. This means that the constructed sloop $SL(20) = 2 \otimes _{\alpha} L_1$ is subdirectly irreducible having only one proper homomorphic image isomorphic to $SL(10)$.

According to the results due to Quackenbush in [12], the variety $V_1$ generated by $SL(10)$ covers the smallest nontrivial subvariety $V_0$ (the class of all Boolean sloops). And according to [2], we may deduce that each of the constructed semi-planar sloop $SL(20) = 2 \otimes _{\alpha} L_1$ generates also a variety $V_1$ (not comparable with $V_1$) covering the variety of all Boolean sloops $V_0$.

**5. Construction an example of each class of $SL(20)$**

Let $(P^*_1, B_1)$ be an STS(9), where $P^*_1 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$ and the set of blocks $B_1$ is given by:

$$B_1 = a_1 a_3 a_4 \quad a_1 a_2 a_6 \quad a_1 a_7 a_8 \quad a_1 a_5 a_9 \quad a_2 a_3 a_9 \quad a_2 a_4 a_7$$
Let \( L_1 = (P_1 = P_1^* \cup \{e\}; \cdot, e) \) be the associated sloop of \((P_1^*; B_1)\). Also, let \( P_2 = \{b, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9\} \). The 1-factorization \( F \) on the set \( P_1 \) and the 1-factorization \( G \) on the set \( P_2 \) are defined as in section 2 by:

\[
F = \{F_1, F_2, \ldots, F_9\}, \text{ where } F_i = \{a_i a_k : a_i \cdot a_k = a_i \text{ and } a_i, a_k \in P\}
\]

\[
G = \{G_1, G_2, \ldots, G_9\}, \text{ where } G_i = \{b b_j \cup \{b_j b_k : a_i \cdot a_k = a_i \text{ for } i \notin \{l, k\}\}
\]

The constructed \( \text{STS}(19) = 2 \otimes_a T_1 \) is defined by \((P^* = P_1^* \cup P_2; B = B_1 \cup B_{12})\), where \( B_{12} = \{\{a_i, b_j, b_k\} : b_j b_k \in G_{a(i)}\} \). The associated sloop \( 2 \otimes_a L_1 = (P = P^* \cup \{e\}; \cdot, e) \) with the \( \text{STS}(19) = 2 \otimes_a T_1 \) has the sub-\( SL(10) = L_1 \) for each permutation \( a \), so \( L_1 \) is always normal in \( 2 \otimes_a L_1 \).

For each block \( \{a_i, a_j, a_k\} \in B_1 \), we have the sub-1-factorizations:

\[
f = \{f_i = \{e a_i, a_j, a_k\}, f_j = \{e a_j, a_i, a_k\}, f_k = \{e a_k, a_i, a_j\}\}
\]

\[
g = \{g_i = \{b b_i, b_j b_k\}, g_j = \{b b_j, b_i b_k\}, g_k = \{b b_k, b_i b_j\}\} \text{ for all } \{i, j, k\} \in X.
\]

Where \( X = \{1, 3, 4\}, \{1, 2, 6\}, \{1, 7, 8\}, \{1, 5, 9\}, \{2, 3, 9\}, \{2, 4, 7\}, \{2, 5, 8\}, \{3, 5, 7\}, \{3, 6, 8\}, \{4, 5, 6\}, \{4, 8, 9\}, \{6, 7, 9\}\) is the set of lines of the affine planar over \( GF(3) \) with the set of points \( N = \{1, 2, \ldots, 9\} \).

By applying the interchange:

\[
H = \{\{a_1, a_3, a_4\}, \{a_1, b_3, b_4\}, \{a_3, b_1, b_4\}, \{a_4, b_1, b_3\}\}
\]

with the set of triples:

\[
R = \{\{b_1, b_3, b_4\}, \{b_1, a_3, a_4\}, \{b_3, a_1, a_4\}, \{b_4, a_1, a_3\}\}
\]

on the set \( A^* = \{a_1, a_3, a_4, b, b_1, b_3, b_4\} \), we will get the associated sloop \( SL(20) = 2 \otimes_a L_1 \) with the constructed triple system \( 2 \otimes_a T_1 = (P^*; B - H \cup R) \).

The following 7 examples supplies us with an example for each class of \( SL(20) \) given in section 3 and in section 4.

Notice that \( \{1, 3, 4\} \) is a line in \( X \). We will choose the permutation \( a \) satisfying that \( a(1) = 1, a(3) = 3 \) and \( a(4) = 4 \) in all examples from (1) to (6):

(1) \( a_1 = \text{id}_N \); i. e., \( a_1 \) transfers each line into the same line in \( X \). The constructed \( SL(20) = 2 \otimes_{a_1} L_1 \) has 12 sub-\( SL(8) \)s and one sub-\( SL(10) \); i. e. \( 2 \otimes_{a_1} L_1 \) is isomorphic to \( SL(10) \times SL(2) \). And the constructed \( 2 \otimes_{a_1} L_1 \) is an \( SL(20) \) having 12 sub-\( SL(8) \)s but no sub-\( SL(10) \). The sloop \( 2 \otimes_{a_1} L_1 \) is subdirectly irreducible having exactly one proper homomorphic image \( \cong SL(10) \).

In all cases (2) – (7), the constructed \( SL(20) = 2 \otimes_{a_1} L_1 \) is subdirectly irreducible. And in all cases (2) - (6), the constructed \( SL(20) = 2 \otimes_{a_1} L_1 \) is a semi-planar sloop.
(2) $\alpha_2 = (26)$; i.e., $\alpha_2$ transfers 6 lines into lines, namely, the set $\{1, 3, 4\}, \{1, 2, 6\}, \{1, 7, 8\}, \{1, 5, 9\}, \{4, 8, 9\}, \{3, 5, 7\}$. The constructed $\text{SL}(20) = 2 \otimes_{\alpha_2} L_1$ has one sub-$\text{SL}(10)$ and 6 sub-$\text{SL}(8)$s. The constructed $\text{SL}(20) = 2 \otimes_{\alpha_2} L_1$ has 6 sub-$\text{SL}(8)$s but no sub-$\text{SL}(10)$.

(3) $\alpha_3 = (26) (78)$; i.e., $\alpha_3$ transfers 4 lines into lines, where the set of the four lines is $\{1, 3, 4\}, \{1, 2, 6\}, \{1, 7, 8\}, \{1, 5, 9\}$. The constructed $\text{SL}(20) = 2 \otimes_{\alpha_3} L_1$ has one sub-$\text{SL}(10)$ and 4 sub-$\text{SL}(8)$s. The constructed $\text{SL}(20) = 2 \otimes_{\alpha_3} L_1$ has 4 sub-$\text{SL}(8)$s but no sub-$\text{SL}(10)$.

(4) $\alpha_4 = (258)$; i.e., $\alpha_4$ transfers 3 lines into lines, where the set of the three lines is $\{1, 3, 4\}, \{2, 5, 8\}, \{6, 7, 9\}$. The constructed $\text{SL}(20) = 2 \otimes_{\alpha_4} L_1$ has one sub-$\text{SL}(10)$ and 3 sub-$\text{SL}(8)$s. The constructed $\text{SL}(20) = 2 \otimes_{\alpha_4} L_1$ has 3 sub-$\text{SL}(8)$s but no sub-$\text{SL}(10)$.

(5) $\alpha_5 = (2567) (89)$; i.e., $\alpha_5$ transfers 2 lines into lines, where the set of lines is $\{1, 3, 4\}, \{4, 8, 9\}$. The constructed $\text{SL}(20) = 2 \otimes_{\alpha_5} L_1$ has one sub-$\text{SL}(10)$ and 2 sub-$\text{SL}(8)$s. The constructed $\text{SL}(20) = 2 \otimes_{\alpha_5} L_1$ has 2 sub-$\text{SL}(8)$s but no sub-$\text{SL}(10)$.

(6) $\alpha_6 = (257968)$; i.e., $\alpha_6$ transfers only the line $\{1, 3, 4\}$ into a line. The constructed $\text{SL}(20) = 2 \otimes_{\alpha_6} L_1$ has only one sub-$\text{SL}(10)$ and one sub-$\text{SL}(8)$. The constructed $\text{SL}(20) = 2 \otimes_{\alpha_6} L_1$ has only one sub-$\text{SL}(8)$ but no sub-$\text{SL}(10)$.

(7) $\alpha_7 = (123456798)$; i.e., $\alpha_7$ transfers no line into a line. The constructed $\text{SL}(20) = 2 \otimes_{\alpha_7} L_1$ has only one sub-$\text{SL}(10)$ and no sub-$\text{SL}(8)$s. In fact, the corresponding $\text{STS}(19) = 2 \otimes_{\alpha_7} T_1$ has exactly one sub-$\text{STS}(9)$, but no sub-$\text{STS}(7)$s. This means that each triangle in the associated squag $\text{SQ}(19)$ either generates the whole $\text{SQ}(19)$ or a sub-$\text{SQ}(9)$. Which implies that the associated squag $\text{SQ}(19)$ is an example of a semi-planar squag of cardinality 19. We note that the smallest known cardinality of semi-planar squag is 21 (cf. [2]).

The subsloops mentioned in the above examples are $\text{SL}(10) = L_1$, in which $L_1$ is always normal in $2 \otimes_{\alpha} L_1$ and the sub-$\text{SL}(8)$s determined by the set $\{e, a_i, a_j, a_k, b, b_{\alpha(i)}, b_{\alpha(j)}, b_{\alpha(k)}\}$, in which $\{i, j, k\}$ and $\{\alpha(i), \alpha(j), \alpha(k)\}$ are lines of the affine plane over $\text{GF}(3)$.

References


Appendix including proofs of lemmas 1, 2, 3 and 4

Proof of lemma 1. We have that $P_1 := \{e, a_1, \ldots, a_n\}$ and $P_2 := \{b, b_1, \ldots, b_n\}$ are disjoint sets having the same cardinality $n$, in which $P_1$ forms a subloop of $2 \otimes_a L_1$. Let $S$ satisfy $S - P_i \neq \emptyset$ for $i = 1$ and 2, then $|S| \geq 2$. If $|S| = 2$, then $S = \{a_i, b_j\}$. For $|S| > 2$, since $a_i \cdot b_k$ are always element of $P_2$, for all $a_i \in P_1$ and $b_k \in P_2$, then $S$ contains at least a 2-element subset $\{a_i, a_j\} \subseteq P_1$ and a 2-element subset $\{b_k, b_l\} \subseteq P_2$. Consider the map $\delta_{b_k}(x) := x \cdot b_k$ for $x \in S \cap P_1$. It is easy to see that the map $\delta_{b_k}$ is bijective from the subset $S \cap P_1$ onto the subset $S - P_1$. Which implies that $|S \cap P_1| = (1/2)|S| = |S \cap P_2|$.

Proof of lemma 2. Let $f$ be a sub-1-factorization on $K_r$ of $F$. Then the order $r$ of the complete graph $K_r$ is an even number less than or equal to $10/2 = 5$, hence $r = 4$. Indeed, if there is a sub-1-factorization on a 4-element subset $\{x, y, z, w\}$, then $e \in \{x, y, z, w\}$. Otherwise, assume that $e \not\in \{x, y, z, w\}$ and $\{x y, z w\} \subseteq F_i, \{x z, y w\} \subseteq F_j$ and $\{x w, y z\} \subseteq F_k$ form a sub-1-factorization of $K_4$. But $e a_i \in F_i, e a_j \in F_j$ and $e a_k \in F_k$, then $\{e, a_i, a_j, a_k\} \cap \{x, y, z, w\} = \emptyset$. This implies that $F_i = \{e a_i, a_i a_k, x y, z w, u v\}$ or $\{e a_i, a_j u, a_i v, x y, z w\}$, Hence the first case of $F_i$ tends to the 1-factor $F_j = \{e a_j, a_i a_k, x z, y w, u v\}$ and the second case of $F_i$ leads to the 1-factor $F_j = \{e a_j, a_k v, x z, y w\}$, both cases contradict the fact that $F_i \cap F_j = \emptyset$. Similarly, if there is a sub-1-factorization on a 4-element subset $\{x, y, z, w\}$ of $P_2$, then $b$ must be an element of $\{x, y, z, w\}$. Since the number of blocks of $B_1$ is 12, each of the 1-factorizations $F$ and $G$ has exactly 12 sub-1-factorizations of $K_4$. This completes the proof.
Proof of lemma 3. Let $2 \otimes \alpha_1 C_I = (C_I \cup C_2; \cdot, e)$ be a subsloop of $2 \otimes \alpha L_1$. Since $C_I = \{e, a_i, a_j, a_k\}$ is a subsloop of $L_1$, there is a sub-1-factorization $f = \{f_i, f_j, f_k\}$ on $C_I$. According to the definition of $2 \otimes \alpha_1 C_I$, there is a sub-1-factorization $g = \{g_{a(i)} = \{b b_{a(i)}, b a_{a(j)} b a_{a(k)}\} \subseteq G_{a(i)}\}$ on the subset $C_2 = \{b, b a_{a(i)}, b a_{a(j)}, b a_{a(k)}\}$ if and only if $\{\alpha(i), \alpha(j), \alpha(k)\}$ is a line in $X$. This implies that $2 \otimes \alpha_1 C_I = (C_I \cup C_2; \cdot, e)$ is a subsloop of $2 \otimes \alpha L_1$, if and only if $\{\alpha(i), \alpha(j), \alpha(k)\}$ is a line in $X$.

Proof of lemma 4. According to Lemma 2, we may say that $S \cap L_1 = C_I = C_1 = \{e, a_i, a_j, a_k\}$ is a 4-element subsloop and $\{i, j, k\}$ is a line in $X$. So there is a sub-1-factorization $f = \{f_i = \{e a_i, a_j a_k, a_j a_k\}, f_j = \{e a_j, a_i a_k\}, f_k = \{e a_k, a_i a_j\}\}$ on $C_I$. According to the construction $2 \otimes \alpha L_1$ we have: $f_i$ related with $G_{a(i)}$, $f_j$ related with $G_{a(j)}$ and $f_k$ related with $G_{a(k)}$. Since $S$ is a sub-$SL(8)$, then $\{\alpha(i), \alpha(j), \alpha(k)\}$ is a line in $X$ and the three 1-factors $G_{a(i)}$, $G_{a(j)}$ and $G_{a(k)}$ contains a sub-1-factorization $g = \{g_{a(i)} = \{b b_{a(i)}, b a_{a(j)} b a_{a(k)}\}, g_{a(j)} = \{b b_{a(j)}, b a_{a(i)} b a_{a(k)}\}, g_{a(k)} = \{b b_{a(k)}, b a_{a(i)} b a_{a(j)}\}\}$ on the 4-element subset $C_2 = \{b, b a_{a(i)}, b a_{a(j)}, b a_{a(k)}\}$ of $P_2$. According to the definition of the set of blocks $B_{12}$ and using the sub-1-factorizations $f$ and $g$, then the subsloop $S$ can be represented by the construction $2 \otimes \alpha_1 C_I = (C_I \cup C_2; \cdot, e)$, where $\alpha_1$ is equal to $\alpha$ restricted on the subset $\{i, j, k\}$. This completes the proof.