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LUCAS' THEOREM FOR EXTENDED GENERALIZED BINOMIAL COEFFICIENTS

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1. Introduction

Lucas' theorem says that for prime p with $n=\sum_{j=0}^{r} n_j p^j$ and $m=\sum_{j=0}^{r} m_j p^j$, where $0 \le \{n_j, m_j\} < p$, we have

$$\binom{n}{m} \equiv \prod_{j=0}^{r} \binom{n_j}{m_j} \pmod{p}.$$

Bollinger and Burchard [1] have derived a corresponding theorem for the generalized binomial coefficients, namely,

$$\binom{n}{m}_s \equiv \sum_{(s_0,\dots,s_r)} \prod_{j=0}^{r} \binom{n_j}{s_j}_s \pmod{p}$$

where $(s_0,\dots,s_r)$ are all $(r+1)$-tuples such that $m=\sum_{j=0}^{r} s_j p^j$ and $0 \le s_j \le q n_j$.

We have shown in [4,5] that the k-extensions of the generalized binomial coefficients have direct combinatorial interpretations and satisfy

$$\sum_{m=0}^{qn} m!^{-b}\,{}^{k}\!\binom{n}{m}_q x^m = \begin{cases} \dfrac{{}^{k}T_q(x)^{n+1}-1}{{}^{k}T_q(x)-1} \quad (= n+1 \text{ for } {}^{k}T_q(x)=1) & k = 0 \text{ to } 3 \\ {}^{k}T_q(x)^n & k = 4 \text{ to } 7 \end{cases} \tag{1.1}$$

where $0 \le k = abc_2 \le 7$ and

$${}^{k}T_q(x) = a + \sum_{i=1}^{q} i!^{c-b} x^i = \sum_{i=1-a}^{q} i!^{c-b} x^i .$$

In particular, $\binom{n}{m}_s = {}^{4}\!\binom{n}{m}_q$ with $q = s-1$. Corresponding extensions to Lucas' theorem are considered in the following sections.

2. Lucas' theorem for extensions *k* = 4 to 7

For k = 4 to 7, the derivation of a Lucas-type theorem is similar to that given by Bollinger and Burchard although we have also implicitly demonstrated its validity for generating functions which are powers of any polynomial. In the following, note that since $a^{p^j} \equiv a \pmod p$, j integer, it can be shown that $P(x)^{p^j} \equiv P(x^{p^j}) \pmod p$ for polynomial $P(x)$.

$$\begin{aligned}
\sum_{m=0}^{qn} m!^{-b} \, {}^{k}\binom{n}{m}_q x^m &= {}^{k}T_q(x)^n \\
&= \prod_{j=0}^{r} {}^{k}T_q(x)^{p^j n_j} \\
&\equiv \prod_{j=0}^{r} {}^{k}T_q(x^{p^j})^{n_j} \pmod p \\
&= \prod_{j=0}^{r} \left\{ \sum_{s_j=0}^{qn_j} s_j!^{-b} \, {}^{k}\binom{n_j}{s_j}_q x^{s_j p^j} \right\} \\
&= \sum_{m=0}^{qn} \left\{ \sum_{(s_0,\dots,s_r)} \prod_{j=0}^{r} s_j!^{-b} \, {}^{k}\binom{n_j}{s_j}_q \right\} x^m
\end{aligned}$$

where $(s_0,\dots,s_r)$ are all (r+1)-tuples such that $m = \sum_{j=0}^{r} s_j p^j$ and $0 \le s_j \le qn_j$. Thus,

$$m!^{-b} \, {}^{k}\binom{n}{m}_q \equiv \sum_{(s_0,\dots,s_r)} \prod_{j=0}^{r} s_j!^{-b} \, {}^{k}\binom{n_j}{s_j}_q \pmod p \qquad (2.1)$$

after equating coefficients of powers of x. For k = 4 and 7, this reverts to Bollinger and Burchard's formula for the generalized binomial coefficients. Examplar values of the coefficients are given in Table 3 of [4].

<u>Example 1</u>: ${}^{5}\binom{6}{5}_2 = 366 \equiv 1 \pmod 5$.

Here, $(a,b,c) = (1,0,1)$, q = 2, n = 6 and m = 5. Thus, $6 = 11_5 \Rightarrow (n_0, n_1) = (1,1)$ and $r = 1$. Also, $s_0 + s_1 5 = 5$ with $0 \le \{s_0, s_1\} \le 2 \cdot 1$ gives $(s_0, s_1) = (0,1)$. The right-hand side of (2.1) is then ${}^{5}\binom{1}{0}_2 \, {}^{5}\binom{1}{1}_2 = 1$ as required.

<u>Example 2</u>: $3!^{-1} \, {}^{6}\binom{5}{3}_2 = 20 \equiv 2 \pmod 3$.

Now $(a,b,c) = (1,1,0)$, q = 2, n = 5 and m =3 so that $5 = 12_3 \Rightarrow (n_0, n_1) = (2,1)$ and $r = 1$. Also, $s_0 + s_1 3 = 3$ with $0 \le s_0 \le 2 \cdot 2$ and $0 \le s_1 \le 2 \cdot 1$ gives $(s_0, s_1) = (3,0)$ or $(0,1)$. The right-hand side of (2.1) is then

$$3!^{-1}\ {}^{6}\binom{2}{3}_2\ 0!^{-1}\ {}^{6}\binom{1}{0}_2 + 0!^{-1}\ {}^{6}\binom{2}{0}_2\ 1!^{-1}\ {}^{6}\binom{1}{1}_2 = 2$$

as required.

3. Results for extensions *k* = 0 to 3

A similar formula for k = 0 to 3 remains an open problem. However, results for certain combinations of the extended coefficients can be obtained as follows. For k = 0 to 3,

$${}^{k}T_q(x) = \sum_{i=1}^{q} i!^{c-b} x^i = x\sum_{i=0}^{q-1}(i+1)!^{c-b}x^i \text{ and } \sum_{m=0}^{qn} m!^{-b}\ {}^{k}\binom{n}{m}_q x^m = \frac{{}^{k}T_q(x)^{n+1}-1}{{}^{k}T_q(x)-1}.$$

Thus,

$$\left(1+\left({}^{k}T_q(x)-1\right)\sum_{m=0}^{qn} m!^{-b}\ {}^{k}\binom{n}{m}_q x^m\right)\Big/{}^{k}T_q(x)={}^{k}T_q(x)^n \tag{3.1}$$

whence we define ${}^{k}\binom{n}{m}_q^{*}$ by

$$\begin{aligned} x^n \sum_{m=0}^{(q-1)n} m!^{-b}\ {}^{k}\binom{n}{m}_{q-1}^{*} x^m &= {}^{k}T_q(x)^n \\ &= \sum_{m=0}^{qn} m!^{-b}\ {}^{k}\binom{n}{m}_q x^m - \left(\sum_{m=1}^{qn} m!^{-b}\ {}^{k}\binom{n}{m}_q x^m\right)\Big/{}^{k}T_q(x). \end{aligned} \tag{3.2}$$

This is a reasonable definition because of the form of ${}^{k}T_q(x)$. Relationships between the asterisked and un-asterisked coefficients can be obtained by equating coefficients of powers of x. Applying the approach used previously for k = 4 to 7 leads to

$$m!^{-b}\ {}^{k}\binom{n}{m}_q^{*} \equiv \sum_{(s_0,\ldots,s_r)} \prod_{j=0}^{r} s_j!^{-b}\ {}^{k}\binom{n_j}{s_j}_q^{*} \pmod p. \tag{3.3}$$

In particular, when k = 0 or 3,

$${}^{k}T_q(x)^n = x^n\ {}^{k+4}T_{q-1}(x)^n = x^n \sum_{m=0}^{(q-1)n} m!^{-b}\ {}^{k+4}\binom{n}{m}_{q-1} x^m$$

so

$${}^{k}\binom{n}{m}_q^{*} = {}^{k+4}\binom{n}{m}_q \tag{3.4}$$

and (3.3) is clearly satisfied because of (2.1). Also, solving (3.2) for the asterisked coefficients and substituting in (3.4) gives further interrelationships between the k-extensions.

Example 3: ${}^{0}\binom{4}{2}_1^{*}$ is the coefficient of x^6 on the LHS of (4). The corresponding coefficient on the right hand side is

$${}^{0}\binom{4}{6}_2 - \sum_{m=1}^{7}(-1)^m\,{}^{0}\binom{4}{m}_2 = 7-(1-2+3-5+7-7+4) = 6\,.$$

(*Mathematica* was used to do the required polynomial division.) From (3.4), ${}^{0}\binom{4}{2}_1^{*} = {}^{4}\binom{4}{2}_1 = \binom{4}{2} = 6$ as required and the coefficient interrelationship is given by

$${}^{4}\binom{4}{2}_1 = {}^{0}\binom{4}{6}_2 - \sum_{m=1}^{7}(-1)^m\,{}^{0}\binom{4}{m}_2\,.$$

In this example, (3.3) effectively reverts to an application of Lucas' theorem for the binomial coefficients. However, to complete the example, the calculations based on (3.2) will be demonstrated for $p=3$. Here $(a,b,c)=(0,0,0)$, $q=1$, $n=4=11_3$ and $m=2=02_3$ so (3.3) becomes

$${}^{0}\binom{4}{2}_1^{*} \equiv {}^{0}\binom{1}{2}_1^{*}\,{}^{0}\binom{1}{0}_1^{*} \pmod 3\,.$$

Equation (3.2) gives

$${}^{0}\binom{1}{2}_1^{*} = {}^{0}\binom{1}{3}_2 - \sum_{m=1}^{2}(-1)^m\,{}^{0}\binom{1}{m}_2 = 0-(-1+1) = 0$$

and so the right hand side is zero as expected.

Finally, equation (3.2) also indicates that terms on the right hand side are zero for $0 \le u < n$. Thus,

$$u!^{-b}\,{}^{k}\binom{n}{u}_q - u!^{-1}\frac{d^u}{dx^u}\left(\frac{\sum_{m=1}^{qn} m!^{-b}\,{}^{k}\binom{n}{m}_q x^m}{{}^{k}T_q(x)}\right)_{x=0} = 0\,.$$

For example, k = 0, q = 2, n = 6 and u = 5 leads to

$${}^{0}\binom{6}{5}_2 - \sum_{m=1}^{6}(-1)^m\,{}^{0}\binom{6}{m}_2 = 8+1-2+3-5+8-13 = 0$$

which clearly exploits a property of the Fibonacci sequence. Similar results (and involving more general sequences) also hold for the other k-extensions, k = 0 to 3, of all orders q.

4. Conclusion

The k-extensions to the generalized binomial coefficients admit results which extend Lucas' theorem. Direct parallels of Bollinger and Burchard's result have been determined for k = 4 to 7.

Related congruences have also been derived for k = 0 to 3 but an exact parallel remains an open problem. Further investigations could also involve relating the extensions of Lucas' theorem to various graphs as in [6] and the extension of other congruence results such as those of Carlitz [2,3].

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