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A NEW DIRECTION OF FIBONACCI SEQUENCE MODIFICATION

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The Fibonacci sequence 0,1,1,2,3,5,... is an object of different modifications and extensions.

In [1,2] four different ways of constructing two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ are described and called *2-Fibonacci sequences* (or *2-F-sequences).* The four schemes are the following

$$\begin{aligned}
&\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d\\
&\alpha_{n+2} = \beta_{n+1} + \beta_n,\ n \geq 0\\
&\beta_{n+2} = \alpha_{n+1} + \alpha_n,\ n \geq 0
\end{aligned}$$

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\end{aligned}$$

Obviously, the Third and the Fourth schemes contain two standard Fibonacci sequences and therefore they are trivial modification, while the first two schemes are essential extensions of Fibonacci sequence.

Graphically, the (n+2)-nd members of the four schemes are obtained from the n-th and the (n+1)-st members as shown in Fig. 1 - 4.

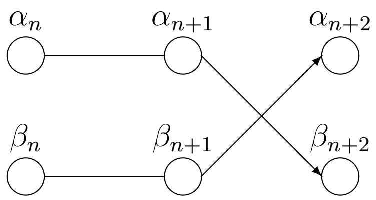

Fig. 1.

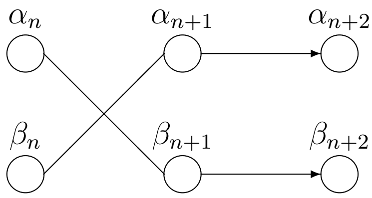

Fig. 2.

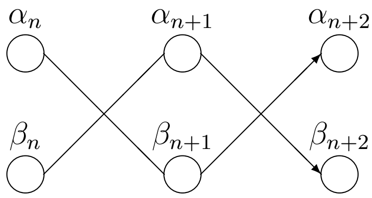

Fig. 3.

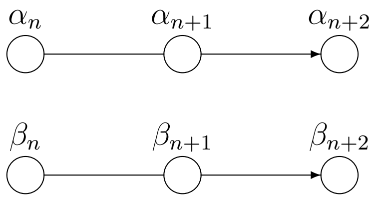

Fig. 4.

Clearly, if we set $a = b$ and $c = d$, then sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ will coincide with each other and with the sequence $\{F_i\}_{i=0}^{\infty}$, which is called a generalized Fibonacci sequence, where

$$F_0(a, c) = a,$$

$$F_1(a, c) = c,$$

$$F_{n+2}(a, c) = F_{n+1}(a, c) + F_n(a, c).$$

Let $F_i = F_i(0, 1)$; $\{F_i\}_{i=0}^{\infty}$ be the ordinary Fibonacci sequence.

The idea for 2-F-sequences are objects of next research (see, e.g., [3-22]). These sequences were extended to 3-Fibonacci sequences and to 2-Tribonacci sequences. The possibility for defining of k-Fibonacci sequences, of k-Tribonacci sequences, and more generally - of k-m-bonacci sequences are mentioned. In all cases the members of the sequences have representations through sums of coeffitients and basic numbers a, b, c, d and they do not contain free terms. Now, we shall introduce new extensions of the 2-Fibonacci sequences which have the following forms.

$$\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d$$
$$\alpha_{n+2} = \beta_{n+1} + \beta_n + p,\ n \geq 0$$
$$\beta_{n+2} = \alpha_{n+1} + \alpha_n + q,\ n \geq 0$$

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The first 10 members of the First scheme have the forms:

α_n	β_n
a	b
c	d
$b+d+p$	$a+c+q$
$a+c+d+p+q$	$b+c+d+p+q$
$a+b+2c+d+2p+2q$	$a+b+c+2d+2p+2q$
$a+2b+2c+3d+4p+3q$	$2a+b+3c+2d+3p+4q$
$3a+2b+4c+4d+6p+6q$	$2a+3b+4c+4d+6p+6q$
$4a+4b+7c+6d+10p+10q$	$4a+4b+6c+7d+10p+10q$
$6a+7b+10c+11d+17p+16q$	$7a+6b+11c+10d+16p+17q$
$11a+10b+17c+17d+27p+27q$	$10a+11b+17c+17d+27p+27q$
$17a+17b+28c+27d+43p+44q$	$17a+17b+27c+18d+44p+43q$
$\dots$	$\dots$

where a, b, c, d are given constants.

THEOREM 1. For each natural number $n \geq 0$

$$\alpha_{n+2} = \frac{1}{2}.((F_{n+1} + 3.[\frac{n+2}{3}] - n - 1).a + (F_{n+1} - 3.[\frac{n+2}{3}] + n + 1).b$$
$$+(F_{n+2} - 3.[\frac{n}{3}] + n - 1).c + (F_{n+2} + 3.[\frac{n}{3}] - n + 1).d)$$
$$+(F_{n+3} + [\frac{n}{3}] - [\frac{n+2}{3}]).p + (F_{n+3} - [\frac{n}{3}] + [\frac{n+2}{3}] - 2).q)$$
$$= \frac{1}{2}.((a+b).F_{n+1} + (c+d).F_{n+2} + (3.[\frac{n+2}{3}] - n - 1).(a-b) + (n - 3.[\frac{n}{3}] - 1).(c-d)$$

$$(p+q).F_{n+3} - ([\frac{n+2}{3}] - [\frac{n}{3}]).(p-q) - 2q),$$

$$\beta_{n+2} = \frac{1}{2}.((F_{n+1} - 3.[\frac{n+2}{3}] + n + 1).a + (F_{n+1} + 3.[\frac{n+2}{3}] - n - 1).b$$

$$+(F_{n+2} + 3.[\frac{n}{3}] - n + 1).c + (F_{n+2} - 3.[\frac{n}{3}] + n - 1).d$$

$$+(F_{n+3} - [\frac{n}{3}] + [\frac{n-1}{3}] - 2).p + (F_{n+3} + [\frac{n}{3}] - [\frac{n+2}{3}]).q)$$

$$= \frac{1}{2}.((a+b).F_{n+1} + (c+d).F_{n+2} + (3.[\frac{n+2}{3}] - n - 1).(b-a) + (n - 3.[\frac{n}{3}] - 1).(d-c)$$

$$+(p+q).F_{n+3} + ([\frac{n+2}{3}] - [\frac{n}{3}]).(p-q) - 2p).$$

The proof of this and the next assertions can be made, for example, by induction.

The first 10 members of the Second scheme have the forms:

α_n	β_n
a	b
c	d
$b+c+p$	$a+d+q$
$b+c+d+2p$	$a+c+d+2q$
$a+b+c+2d+3p+q$	$a+b+2c+d+p+3q$
$2a+b+2c+3d+4p+3q$	$a+2b+3c+2d+3p+4q$
$3a+2b+4c+4d+6p+6q$	$2a+3b+4c+4d+6p+6q$
$4a+4b+7c+6d+10p+10q$	$4a+4b+6c+7d+10p+10q$
$6a+7b+11c+10d+17p+16q$	$7a+6b+10c+11d+16p+17q$
$10a+11b+17c+17d+28p+26q$	$11a+10b+17c+17d+26p+28q$
$17a+17b+27c+28d+45p+43q$	$17a+17b+28c+27d+43p+45q$
$\dots$	$\dots$

where a, b, c, d are given constants.

Let ψ be the integer function defined for every $k \geq 0$ by:

r	$\psi(6.k+r)$
0	1
1	0
2	-1
3	-1
4	0
5	1

Obviously, for every $n \geq 0$,

$$\psi(n+3) = -\psi(n).$$

Using the definition of the function ψ, the following assertions are proved by induction.

THEOREM 2. If $n \geq 1$, then

$$\alpha_n = \frac{1}{2}.((F_{n-1} + \psi(n)).a + (F_{n-1} + \psi(n+3)).b + (F_n + \psi(n+4)).c + (F_n + \psi(n+1)).d$$

$$+(F_{n+1} + \psi(n+2)).p + (F_{n+1} - 2 - \psi(n+2)).q$$

$$= \frac{1}{2}.((a+b).F_{n-1} + (c+d).F_n + (p+q).F)n + 1 + \psi(n).a + \psi(n+3).b + \psi(n+4).c + \psi(n+1).d$$

$$+(p-q).\psi(n+2) - 2q),$$

$$\beta_n = \frac{1}{2}.((F_{n-1} + \psi(n+3)).a + (F_{n-1}\psi(n).b + (F_n + \psi(n+1)).c + (F_n + \psi(n+4)).d)$$

$$+(F_{n+1} - 2 - \psi(n+2)).p + (F_{n+1} + \psi(n+2)).q$$

$$= \frac{1}{2}.((a+b).F_{n-1} + (c+d).F_n + (p+q).F)n + 1 + \psi(n+3).a + \psi(n).b + \psi(n+1).c + \psi(n+4).d$$

$$-(p-q).\psi(n+2) - 2p).$$

The first 10 members of the Third scheme have the forms:

α_n	β_n
a	b
c	d
$a+d+p$	$b+c+q$
$b+2c+p+q$	$a+2d+p+q$
$2a+3d+3p+q$	$2b+3c+p+3q$
$3b+5c+3p+4q$	$3a+5d+4p+3q$
$5a+8d+8p+4q$	$5b+8c+4p+8q$
$8b+13c+8p+12q$	$8a+13d+12p+8q$
$13a+21d+21p+12q$	$13b+21c+12p+21q$
$21a+34d+21p+33q$	$21a+34d+33p+21q$
$\dots$	$\dots$

where a, b, c, d are given constants.

THEOREM 3. For each natural number $n \geq 0$

$$\alpha_{2n+2} = F_{2n+1}.a + F_{2n+2}.d + F_{2n+2}.p + (F_{2n+1} - 1).q,$$

$$\beta_{2n+2} = F_{2n+1}.b + F_{2n+2}.c + (F_{2n+1} - 1).p + F_{2n+2}.q,$$

$$\alpha_{2n+3} = F_{2n+2}.b + F_{2n+3}.c + F_{2n+2}.p + (F_{2n+3} - 1).q,$$

$$\beta_{2n+3} = F_{2n+2}.a + F_{2n+3}.d + (F_{2n+3} - 1).p + F_{2n+2}.q.$$

The first 10 members of the Fourth scheme have the forms:

α_n	β_n
a	b
c	d
$a+c+p$	$b+d+q$
$a+2c+2p$	$b+2d+2q$
$2a+3c+4p$	$2b+3d+4q$
$3a+5c+7p$	$3b+5d+7q$
$5a+8c+12p$	$5b+8d+12q$
$8a+13c+20p$	$8b+13d+20q$
$13a+21c+33p$	$13b+21d+33q$
$21a+34c+54p$	$21b+34d+54q$
$\dots$	$\dots$

where a, b, c, d are given constants.

THEOREM 4. For each natural number $n \geq 0$

$$\alpha_{n+2} = F_{n+1}.a + F_{n+2}.c + (F_{n+3} - 1).p,$$

$$\beta_{n+2} = F_{n+1}.b + F_{n+2}.d + (F_{n+3} - 1).q.$$

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