NNTDM 12 (2006) 1, 21-24

PROPERTIES OF THE SÀNDOR FUNCTION

GABRIEL MINCU AND LAURENŢIU PANAITOPOL

ABSTRACT. For $x > 0$ one define the function $S(x) = \min\{m \in \mathbb{N} | x \leq m!\}$. We prove that for $x > \sqrt{13!}$ the interval $(S(x), S(x^2))$ contains at least a prime number and that for real $x, y > 0$ the inequality $S(x) + S(y) \geq S(xy)$ holds true. We also study the convergence of a couple of number series involving $S(x)$.

J. Sandor introduced in [3] for $x > 0$ the function $S(x) = \min\{m \in \mathbb{N} | x \leq m!\}$, about which he proved that

$$S(x) \sim \frac{\log x}{\log \log x}. \tag{1}$$

The consequence $S(x^2) \sim 2S(x)$ of (1) together with the Bertrand-Tchebychev theorem suggest us the following Proposition:

Proposition 1. *For all real* $x > \sqrt{13!} \cong 78911.47445$*, the interval* $(S(x), S(x^2))$ *contains at least a prime number.*

For the proof of this Proposition, we will need the following Lemma:

Lemma 2. *a) For all integers* $n \geq 2$ *we have*

$$\log(n-1)! > n \log n - n - \log n. \tag{2}$$

b) For all integers $n \geq 13$ *we have*

$$\log n! < n \log n - 0.83n. \tag{3}$$

Proof. We will prove both relations using induction on n.
a) Relation (2) obviously checks out for $n = 2$. If we suppose it true for $n \geq 2$, we obtain

$$\log n! > n \log n - n. \tag{4}$$

on the other hand, the well-known relation $1 > \log\left(1 + \frac{1}{n}\right)^n$ implies

$$n \log n - n > (n+1)\log(n+1) - (n+1) - \log(n+1); \tag{5}$$

Combining (4) with (5), we get $\log n! > (n+1)\log(n+1) - (n+1) - \log(n+1)$, so (2) is still true for $n + 1$.
b) Computer checking shows that (3) is true for $n = 13$. If we suppose it true for $n \geq 13$, we obtain

$$\log(n+1)! < n \log n - 0.83n + \log(n+1). \tag{6}$$

Key words and phrases. Sandor function, prime numbers, inequalities, series.
2000*Mathematics Subject Classification*:11A25,11A41.

Now, using the inequality $\log\left(1+\frac{1}{n}\right)^n > 0.83$ (which holds for every $n \geq 3$), we get

$$n\log n - 0.83n + \log(n+1) < (n+1)\log(n+1) - 0.83(n+1). \tag{7}$$

Relations (6) and (7) give $\log(n+1)! < (n+1)\log(n+1) - 0.83(n+1)$, so (3) is still true for $n+1$, and we are done. □

Proof of Proposition 1.
Let us denote $m = S(x)$ and $n = S(x^2)$.
We first consider the case $m \geq 13$. Since $(m-1)! < x \leq m!$ and $(n-1)! < x^2 \leq n!$, we obtain $n! > ((m-1)!)^2$, so $\log n! > 2\log(m-1)!$. Using Lemma 2, we get the inequality

$$n(\log n - 0.83) > 2(m\log m - m - \log m). \tag{8}$$

Rohrbach and Weiss showed in [2] that for any integer $n \geq 118$ we can find a prime $p \in \left(n, \frac{14}{13}n\right)$. Direct checking for the values $n < 118$ shows that for every $m \geq 9$ we can find a prime $p \in \left(m, \frac{4}{3}m\right)$. If we suppose that $n < \frac{4m}{3}$, relation (8) implies

$$\frac{4m}{3}\left(\log m + \log\frac{4}{3} - 0.83\right) > 2(m\log m - m - \log m), \tag{9}$$

so

$$2(\log\frac{4}{3} - 0.83) + 3 + 3\frac{\log m}{m} > \log m. \tag{10}$$

For $m \geq 13$, we have $3\frac{\log m}{m} < 0.5919114$; using (10), we get $\log m < 2.508$, leading to the contradiction $m < 12.29 < 13$.
Thus, for $m \geq 13$ we have $n \geq \frac{4m}{3}$. Since we noticed above that in the given conditions we can find a prime $p \in \left(m, \frac{4}{3}m\right)$, it follows that if $S(x) \geq 13$, the interval $(S(x), S(x^2))$ contains at least a prime number.
If $x \in (\sqrt{13!}, 13!]$ then $x^2 > 13!$, so $13 \in (S(x), S(x^2))$.
Finally, if $x = \sqrt{13!} = 1440\sqrt{3003}$, then $S(x) = 9$ and $S(x^2) < 13$, so the interval $(S(x), S(x^2))$ contains no prime number. □

We now study two series involving $S(x)$:

Proposition 3. *a) The series*

$$\sum_{n=1}^{\infty} \frac{S(n+1) - S(n)}{n} \tag{11}$$

is convergent and its sum is e.
b) The series

$$\sum_{n=1}^{\infty} \frac{S(2n) - S(n)}{n} \tag{12}$$

is divergent.

Proof. a) Since we may write

$$S(n+1) - S(n) = \begin{cases} 2, & \text{if } n = 1 \\ 1, & \text{if } n = k!, \quad k \geq 2 \\ 0, & \text{otherwise} \end{cases},$$

we get

$$\sum_{n=2}^{\infty} \frac{S(n+1) - S(n)}{n} = \sum_{k=2}^{\infty} \frac{1}{k!} = e - 2,$$

so

$$\sum_{n=1}^{\infty} \frac{S(n+1) - S(n)}{n} = 2 + \sum_{n=2}^{\infty} \frac{S(n+1) - S(n)}{n} = e.$$

b) For $n \geq 2$ we have

$$S(2n) - S(n) \leq 1. \tag{13}$$

Let $m = S(n)$. If we require equality to hold in (13), we must have $(m-1)! < n \leq m!$ and $m! < 2n \leq (m+1)!$. Therefore, for a fixed m, the values of n for which $S(2n) - S(n) = 1$ and $S(n) = m$ are

$$\frac{1}{2}m! + 1, \frac{1}{2}m! + 2, ..., m!. \tag{14}$$

Consequently, if we associate to each $m \geq 2$ the sum

$$s_m = \frac{1}{\frac{1}{2}m! + 1} + \frac{1}{\frac{1}{2}m! + 2} + ... + \frac{1}{m!}, \tag{15}$$

we may write

$$\sum_{n=2}^{\infty} \frac{S(2n) - S(n)}{n} = \sum_{m=2}^{\infty} s_m. \tag{16}$$

Since s_m has $\frac{1}{2}m!$ terms, it follows that $s_m \geq \frac{1}{2}$; using (16), case b) follows. □

Another interesting property of $S(x)$ is given in

Proposition 4. *For all real numbers $x, y > 0$, $S(xy) \leq S(x) + S(y)$.*

Proof. Let $S(x) = m$ and $S(y) = n$. Since $m! \geq x$ and $n! \geq y$, we get $m!n! \geq xy$. Since

$$\frac{(m+n)!}{m!n!} = \binom{m+n}{n} \geq 1,$$

it follows that $(m+n)! \geq m!n! \geq xy$. If we put $t = S(xy)$, t will be the least s such as $s! \geq xy$, so $m + n \geq t$, giving $S(x) + S(y) \geq S(xy)$. □

Remarks:
1. The function S also has other interesting properties; for example, if one denotes by $\omega(n)$ the number of prime factors of n, one can easily show that $S(n) > \omega(n)$ for all $n \geq 2$.
2. The generalised S function (see [1]) is likely to have similar properties.

References

[1] C. Adiga and K. Taekyn *On a generalisation of the Sàndor function.* Proc. of the Jangjian Math.Soc. No.2 (2002), 121-124.
[2] H. Rohrbach and J. Weiss *Zum finiten Fall des Bertrandschen Postulats.* J. Reinen Angew. Math. **214/215** (1964), 432-440.
[3] J. Sàndor, *An additive analogue of the function S.* Notes on Number Theory and Discrete Mathematics **7**, no.2 (2001), 91-95.

Faculty of Mathematics
Str. Academiei 14
RO-010014 Bucharest,
Romania
gamin@al.math.unibuc.ro

Faculty of Mathematics
Str. Academiei 14
RO-010014 Bucharest,
Romania
pan@al.math.unibuc.ro