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On The Sum of Equal Powers of the First n Terms of an Arbitrary Arithmetic Progression (II)

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Let $a_1, a_2, ..., a_n$ be an arbitrary arithmetic progression and d is the difference of the progression, i.e $d = a_2 - a_1 = a_3 - a_2 = ...$

First we define:

$$a_0 = a_1 - d, a_{-1} = a_0 - d, a_{-2} = a_{-1} - d,$$

etc.

Hence a double sequence is constructed below, that one may call hyperarithmetic progression:

$$..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...$$

Second we introduce a symbol $[R(n)]_{n=0}^{n=n}$ which may be applied to an arbitrary arithmetic function $R(n)$ and is defined as:

$$[R(n)]_{n=0}^{n=n} := R(n) - R(0)$$

For example,

$$[a_n a_{n+1} a_{n+2}]_{n=0}^{n=n} = a_n a_{n+1} a_{n+2} - a_0 a_1 a_2,$$

since every sequence is an arithmetic function.

Let $k \geq 1$ and $n \geq 1$ be integers.

The first result of this paper is

Lemma1. It is fulfilled

$$\sum_{i=1}^{n} a_i a_{i+1} \ldots a_{i+k-1} = \frac{[a_n a_{n+1} \ldots a_{n+k}]_{n=0}^{n=n}}{(k+1)d} \tag{1}$$

A modification of the above result is

Lemma2. It is fulfilled

$$\sum_{i=1}^{n} a_{i-k} a_{i-k+1} \ldots a_{i-1} a_i a_{i+1} \ldots a_{i+k} = \frac{[a_{n-k} a_{n-k+1} \ldots a_{n+k+1}]_{n=0}^{n=n}}{(2k+2)d} \tag{2}$$

One may prove these two lemmas by induction.

Let

$$S_k(n) := a_1^k + a_2^k + \ldots + a_n^k \tag{3}$$

The main result of the paper is:

Theorem. The relation

$$\sum_{m=0}^{k} (-1)^{k-m} T(k;m) d^{2(k-m)} S_{2m+1}(n) = \frac{[a_{n-k} a_{n-k+1} \ldots a_{n+k+1}]_{n=0}^{n=n}}{(2k+2)d}, \tag{4}$$

holds, where the numbers $T(k;m)$ are defined by the equality

$$\sum_{m=0}^{k} T(k;m) x^m = (x+1^2)(x+2^2)\ldots(x+k^2) \tag{5}$$

Proof. Let us put

$$f_k(x) := (x-1^2)(x-2^2)\ldots(x-k^2) \tag{6}$$

According to (5) we have

$$(-1)^k f_k(-x) = (x+1^2)(x+2^2)\ldots(x+k^2) = \sum_{m=0}^{k} T(k;m) x^m$$

Hence

$$f_k(x) = \sum_{m=0}^{k} (-1)^{k+m} T(k;m) x^m = \sum_{m=0}^{k} (-1)^{k-m} T(k;m) x^m \tag{7}$$

Further we obtain

$$a_{i-k}a_{i-k+1}...a_{i-1}a_ia_{i+1}...a_{i+k} =$$

$$= a_i(a_{i-1}a_{i+1})(a_{i-2}a_{i+2})...(a_{i-k}a_{i+k})$$

$$= a_i(a_i - 1d)(a_i + 1d)(a_i - 2d)(a_i + 2d)...(a_i - kd)(a_i + kd)$$

$$= a_i(a_i^2 - 1^2d^2)(a_i^2 - 2^2d^2)...(a_i^2 - k^2d^2)$$

$$= d^{2k}a_i(x_i - 1^2)(x_i - 2^2)...(x_i - k^2) = d^{2k}a_if_k(x_i),$$

where

$$x_i = (\frac{a_i}{d})^2$$

Now using (7), for the case $x = x_i = (\frac{a_i}{d})^2$, we obtain

$$a_{i-k}a_{i-k+1}...a_{i+k} = \sum_{m=0}^{k}(-1)^{k-m}T(k;m)d^{2(k-m)}a_i^{2m+1} \qquad (8)$$

Applying $\sum_{i=1}^{m}$ to both hand-sides of (8) and keeping in mind (2) and (3) we receive exactly (4).

The Theorem is proved.

This Theorem shows us that if we want to calculate $S_{2k+1}(n)$ we must know only the numbers $S_1(n), S_3(n), S_5(n), ..., S_{2k-1}(n)$. None of the numbers $S_2(n), , S_4(n), , S_6(n), ...$ is required for that purpose. In [1] J. Riordan introduced numbers that are similar to $T(k;m)$ but are not the same. These numbers are called there central factorial numbers. Of course, one may express $T(k;m)$ using central factorial numbers.

Our investigation here is a continuation of [2]

References

[1] Riordan J., *Combinatorical Identities*,John Wiley & Sons,Inc.,New York-London-Sydney, 1968