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On The Sum of Equal Powers of the First n Terms of an Arbitrary Arithmetic Progression (I)

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1 Introduction

In the paper an explicit formula for the sum

$$S_k(n) := a_1^k + a_2^k + ... + a_n^k \tag{1}$$

is proposed, where $n \geq 2$ and $k \geq 1$ are integers and the sequence $a_1, a_2, ...$ is an arbitrary arithmetic progression. An interesting fact is that this formula has the form:

$$S_k(n) = \frac{f_k(a_{n+1}, a_n) - f_k(a_1, a_0)}{(k+1)d}, \tag{2}$$

where: d is the difference of the progression; $a_0 := a_1 - d$ and $f_k(x, y)$ is a symmetric polynomial of two variables. The degree of this polynomial is equal to $k+1$. The coefficients of $f_k(x, y)$ depend on the classical Bernoulli numbers and on the binomial coefficients. The case $k = 1$ provides a new formula for the sum of the first n terms of an arbitrary arithmetic progression.

Another interesting fact is that the right hand-side of (2) admits a represantation in symmetric form with respect to $x = a_{n+1}, y = a_n$, after the introduction of an appropriate symbol.

2 Definition and Some Useful Properties of Bernoulli Numbers and Bernoulli Polynomials

Bernoulli numbers are denoted by $B_n, n = 0, 1, 2, ...$ and are introduced by:

$$B_0 = 1 \tag{3}$$

and

$$\sum_{p=0}^{m}\binom{m+1}{p}B_p = 0, m = 1,2,... \tag{4}$$

Using (3) and (4) one may calculate that:

$$B_1 = \frac{-1}{2}; B_2 = \frac{1}{6}, B_4 = \frac{-1}{30}, B_6 = \frac{1}{42}, B_8 = \frac{-1}{30}, B_{10} = \frac{5}{66}, ... \tag{5}$$

It is a well-known fact that:

$$B_{2t+1} = 0, t \geq 1 \tag{6}$$

The relation (4) admits the *umbral calculus* representation:

$$(1+B)^{m+1} - B^{m+1} = 0, m = 1,2,3...$$

Bernoulli polynomials are introduced by:

$$B_m(z) := \sum_{p=0}^{m}\binom{m}{p}B_p z^{m-p}, \tag{7}$$

or in *umbral calculus* form by

$$B_m(z) := (z+B)^m, m \geq 1$$

Obviously we have

$$B_m(0) = B_m, m \geq 1$$

It is known that the relation:

$$B_m(z) = (-1)^m B_m(1-z), m \geq 1 \tag{8}$$

holds. The *umbral calculus* form of this relation is

$$(z+B)^m = (z-1-B)^m, m \geq 1 \tag{9}$$

All of the facts mentioned above about Bernoulli numbers and Bernoulli polynomials may be found in [1], [2], [3].

3 The Polynomial $f_k(x,y)$ and Some of Its Properties

The polynomial $f_k(x,y)$ is introduced by

$$f_k(x,y) := \sum_{p=1}^{k}\binom{k+1}{p}B_p\sum_{t=1}^{p}(-1)^t\binom{p}{t}x^{k+1-t}y^t, \tag{10}$$

where $k \geq 1$ is an integer and $B_p, p = 0, 1, 2...$ are the Bernoulli numbers. From the above definition $f_k(x, y)$ is a polynomial of two variables and $deg f_k(x, y) = k+1$, where deg denotes the degree of the polynomial.

Lemma1. The recurrent relation

$$f_k(x, y) = B_k(x-y)^k + x f_k(x, y) + \frac{1}{\mathrm{k}+1}(y-x)\frac{\partial}{\partial y} f_k(x, y),$$

holds.

Proof. We have

$$f_k(x, y) = x^{k+1} + \sum_{p=1}^{k} \binom{k+1}{p} B_p(x-y)^p x^{k+1-p}$$

$$= x^{k+1} + \sum_{p=1}^{k} (\binom{k}{p} + \binom{k}{p-1}) B_p(x-y)^p x^{k+1-p}$$

$$= x^{k+1} + \sum_{p=1}^{k-1} \binom{k}{p} B_p(x-y)^p x^{k+1-p} + B_k(x-y)^k + \sum_{p=1}^{k} \binom{k}{p-1} B_p(x-y)^p x^{k+1-p}$$

$$= B_k(x-y)^k + x f_{k-1}(x, y) + J,$$

where

$$J := \sum_{p=1}^{k} \binom{k}{p-1} B_p(x-y)^p x^{k+1-p}$$

It is clear that

$$J = \frac{1}{k+1} \sum_{p=1}^{k} \binom{k+1}{p} B_p p(x-y)^p x^{k+1-p}$$

On the other hand we have:

$$(y-x)\frac{\partial}{\partial y} f_k(x, y) = \sum_{p=1}^{k} \binom{k+1}{p} B_p p(x-y)^p x^{k+1-p}$$

The last two equalities yield:

$$J = \frac{1}{k+1}(y-x)\frac{\partial}{\partial y} f_k(x, y)$$

The above equality and the first equality from the proof provide that the assertion of Lemma1 is proved.

An important property of $f_k(x, y)$ that we will use further is the following.

Lemma2. $f_k(x, y)$ is a symmetric polynomial, i.e.

$$f_k(x, y) = f_k(y, x) \tag{11}$$

Proof. First we observe that $f_k(x, y)$ admits the representation

$$f_k(x, y) = x^{k+1} + \sum_{p=1}^{k} \binom{k+1}{p} B_p(x-y)^p x^{k+1-p}, \tag{12}$$

which follows from (3) and (4) for $m = k + 1$. We rewrite (12) in the form

$$f_k(x,y) = \sum_{p=0}^{k+1} \binom{k+1}{p} B_p(x-y)^p x^{k+1-p} - B_{k+1}(x-y)^{k+1} \tag{13}$$

Let $x = y$. In this case (11) is true.
Let $x \neq y$. We put in (13)

$$z = \frac{\mathrm{x}}{\mathrm{x} - \mathrm{y}}$$

and using (7) we obtain

$$f_k(x,y) = (x-y)^{k+1} B_{k+1}(z) - B_{k+1}(x-y)^{k+1} \tag{14}$$

Now (9) and (14) yield

$$f_k(x,y) = (x-y)^{k+1}(-1)^{k+1} B_{k+1}(1-z) - B_{k+1}(x-y)^{k+1}$$

$$= (x-y)^{k+1}(-1)^{k+1} B_{k+1}(-\frac{x}{x-y}) - B_{k+1}(x-y)^{k+1}$$

$$= \sum_{p=0}^{k+1} \binom{k+1}{p} B_p(y-x)^p y^{k+1-p} - B_{k+1}(x-y)^{k+1}$$

$$= \sum_{p=0}^{k+1} \binom{k+1}{p} B_p(y-x)^p y^{k+1-p} - B_{k+1}(y-x)^{k+1} = f_k(y,x)$$

and Lemma2 is proved.
At the end of the proof we used that

$$B_{k+1}(x-y)^{k+1} = B_{k+1}(y-x)^{k+1}$$

The above equality is true if $k+1$ is odd, because of (6). If $k+1$ is even, then

$$(x-y)^{k+1} = (y-x)^{k+1}$$

and the equality holds again.

4 The Main Result Of The Paper

Now we are ready to formulate and prove the main result of this paper. Let $a_1, a_2, \dots$ be an arbitrary arithmetic progression and d- the difference of that progression.We put

$$a_0 := a_1 - d$$

Theorem. For $S_k(n)$introduced by (1) the representation (2) holds, where $f_k(x,y)$ is given by (10).
Proof. We will use the well-known Euler-Maclaurin formula

$$f(1)+f(2)+...+f(n)=\frac{B_0}{0!}(F(n+1)-F(1))+\frac{B_1}{1!}(f(n+1)-f(1))$$

$$+\frac{B_2}{2!}(f^{(1)}(n+1)-f^{(1)}(1))+...+\frac{B_k}{k!}(f^{(k-1)}(n+1)-f^{(k-1)}(1)), \quad (15)$$

where: $F(x)$ is a polynomial and $degF(x)=k+1$; $f(x)=\frac{d}{dx}F(x)$; $f^{(t)}(x):=(\frac{d}{dx})^t f(x), t=1,2...,k-1$.

Putting

$$F(x)=\frac{(a_1+(x-1)d)^{k+1}}{(k+1)d}$$

we obtain

$$f(x)=(a_1+(x-d)d)^k$$

and (15) after elementary computations yields:

$$S_k(n)=\frac{\sum_{p=0}^{k}\binom{k+1}{p}B_p d^p(a_{n+1}^{k+1-p}-a_1)^{k+1-p}}{(k+1)d}, \quad (16)$$

Now we use that

$$d=a_{n+1}-a_n; d=a_1-a_0$$

Using the last two equalities we may rewrite (16) in the form

$$S_k(n)=I_1-I_2, \quad (17)$$

where

$$(k+1)dI_1:=a_{n+1}^{k+1}+\sum_{p=1}^{k}\binom{k+1}{p}B_p(a_{n+1}-a_n)^p a_{n+1}^{k+1-p} \quad (18)$$

and

$$(k+1)dI_2:=a_1^{k+1}+\sum_{p=1}^{k}\binom{k+1}{p}B_p(a_1-a_0)^p a_1^{k+1-p} \quad (19)$$

From (12) the right hand-side of (18) equals to $f_k(a_{n+1},a_n)$ and the right hand-side of (19) equals to $f_k(a_1,a_0)$. The last observation and (17) prove the Theorem.
Corollary. $S_k(n)$ admits a second representation of the form

$$S_k(n)=\frac{g_k(a_{n+1},a_n)-g_k(a_1,a_0)}{d}, \quad (20)$$

where

$$g_k(x,y):=\sum_{p=0}^{k-1}\binom{k}{p}B_{p+1}\sum_{t=0}^{p}(-1)^{t+1}\frac{1}{t+1}\binom{p}{t}x^{k-t}y^{t+1} \quad (21)$$

Now let us define for an arbitrary arithmetic function $R(n)$ the symbol

$$[R(n)]_{n=0}^{n=n} := R(n) - R(0), \tag{22}$$

Using (22) we may rewrite (2) in the form:

$$S_k(n) = \frac{[f_k(x = a_{n+1}, y = a_n)]_{n=0}^{n=n}}{(k+1)d} \tag{23}$$

In the same way we may rewrite (20) in the form

$$S_k(n) = \frac{[g_k(x = a_{n+1}, y = a_n)]_{n=0}^{n=n}}{d}$$

In the numerator of (23), inside of the symbol defined in (22), we have a symmetric expression with respect to $x = a_{n+1}$ and $y = a_n$

5 Examples and Applications

For $k = 1, 2, 3, 4, 5$ from our formula (2) and from (23) we obtain after some calculations:

$$S_1(n) := a_1 + a_2 + ... + a_n = \frac{a_{n+1}a_n - a_1 a_0}{2d}$$

$$S_2(n) := a_1^2 + a_2^2 + ... + a_n^2 = \frac{a_{n+1}a_n(a_{n+1} + a_n)}{6d}$$

$$S_3(n) := a_1^3 + a_2^3 + ... + a_n^3 = \frac{a_{n+1}^2 a_n^2 - a_1^2 a_0^2}{4d}$$

$$S_4(n) := a_1^4 + a_2^4 + ... + a_n^4 = \frac{[a_{n+1}a_n(-a_{n+1}^3 + 4a_{n+1}^2 + 4a_n^2 - a_n^3)]_{n=0}^{n=n}}{30d}$$

$$S_5(n) := a_1^5 + a_2^5 + ... + a_n^5 = \frac{[(a_{n+1}a_n)^2(6a_{n+1}a_n - (a_{n+1} - a_n)^2)]_{n=0}^{n=n}}{12d}$$

The first of the above five representations is a new formula for the sum of the first n terms of the arithmetic progression $a_1, a_2, ..., a_n, ...$

6 Final Remark

It is not difficult to see that $f_k(x, y)$ admits also the following representation

$$f_k(x, y) = \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k+1}{\alpha} (\sum_{p=\alpha}^{k} \binom{k+1-\alpha}{p-\alpha} B_p) x^{k+1-\alpha} y^\alpha$$

Also $g_k(x,y)$ admits the representations:

$$g_k(x,y) = \sum_{p=1}^{k} \binom{k}{p-1} \frac{1}{p} B_p \sum_{t=1}^{p} (-1)^t \binom{p}{t} x^{k+1-t} y^t;$$

$$g_k(x,y) = \sum_{p=1}^{k} \binom{k}{p-1} B_p \sum_{t=1}^{p} (-1)^t \frac{1}{t} \binom{p-1}{t-1} x^{k+1-t} y^t$$

References

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[2] Abramowitz M., Stegun A., *Handbook of mathematical functions*, National Bureau of Standarts Applied Mathematics series. 55, Issued June 1964

[3] Ireland K.,Rosen M., *A Classical Introduction to Modern Number Theory*, Springer-Verlag, New York, 1982.