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INFINITE SERIES AND MODULAR RINGS

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Thirteen convergent infinite series have been analysed in terms of modular rings. This enables one to assess the contribution of different categories of integers to the infinite series. One class of even integers contributes $\frac{1}{6}\left(\frac{\pi}{4}\right)^2$ to a zeta-function with exponent 2. Another class of even integers makes one quarter the contribution of all the odd integers to this series.

1. Introduction

The concept of infinity has intrigued philosophers and mathematicians for thousands of years with questions such as "how can we add an infinity of quantities and arrive at a finite answer?" In fact, many such convergent infinite series have been developed.

Probably the one of most current interest to professionals and amateurs alike is the zeta-function:

$$\varsigma(n) = \sum_{x=1}^{\infty} \frac{1}{x^n},\ n > 1. \tag{1.1}$$

In particular, the complex zeros of the Riemann zeta-function are presumed to induce the local variations in the distributions of the primes. q-generalizations of the zeta function have been explored in the context of enrichment work by Kim *et al* [1]. Hence, this zeta-function has received an enormous amount of attention in order to prove the Riemann Hypothesis which is that all the nontrivial zeros (the values of n other than -2,-4,-6,…) of the zeta-function have real part ½; that is, the values of s other than -2,-4,-6,… such that $\varsigma(s) = 0$ all lie on the critical line $\sigma = \boldsymbol{R}(s) = \frac{1}{2}$.

"Riemann's 'hypothesis' is the most tantalizing of the unsolved problems of mathematics" [2]. It is the number one problem for the 21st century according to Smale [3].

Riemann [4] developed a clever method for connecting the distribution of primes to properties of the function $\varsigma(s)$. Apostol [5] has an introduction to the relevant analytic number theory, and Edwards [6] has an exposition of some of the early large-scale calculation attacks on the problem.

In this paper we use modular rings to describe some of these series in terms of integer structure so that the composition of the series can be analysed. This opens up exer-

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cises and projects for both secondary and tertiary students of mathematics, especially those preparing to become mathematics teachers.

The approach we take is somewhat analogous to that of Effinger, Hicks and Mullen [7] who "contrast and compare the ring of integers and the ring of polynomials in a single variable over a finite field". The notation we adopt is based on the classic text of Hillman and Alexanderson [8].

2. Modular Rings

Two rings will be used here, namely Z_4 and Z_6, which contain four and six classes respectively [7,8].

$\underline{Z_4}$

The integers in this modular ring may be represented by $4r_i + \bar{i}$, in which $\bar{i}$ is the class and r_i can be considered as the row in an array with the four classes as columns. Even integers occur in Classes $\bar{0}_4$ $(4r_0)$ and $\bar{2}_4$ $(4r_2 + 2)$ with $r_0 r_2 = 0,1,2,3,...$. There are no powers in Class $\bar{2}_4$. Odd integers occur in Classes $\bar{1}_4$ $(4r_1 + 1)$ and $\bar{3}_4$ $(4r_3 + 3)$ with $r_1, r_3 = 0,1,2,3,...$. There are no even powers in Class $\bar{3}_4$. Characteristics of the squares of the odd integers are given in Table 1 [9,10].

	N^2	Class of N	Function for N	Parity of n
Z_4	$12n(3n+1)+1$ $n = 0,1,2,3,...$ $\bar{1}_4$	$\bar{1}_4$ $\bar{3}_4$	$1+12t$ $7+12t$ $t = 0,1,2,3,...$	*even* *odd*
	$12n(3n-1)+1$ $n = 1,2,3,...$ $\bar{1}_4$	$\bar{1}_4$ $\bar{3}_4$	$5+12t$ $11+12t$	*odd* *even*
	$4(1+3n)(2+3n)+1$ $n = 0,1,2,3,...$ $\bar{1}_4$	$\bar{1}_4$ $\bar{3}_4$	$3 \mid N$ $3 \mid N$	*odd* *even*
Z_6	$12n(3n+1)+1$ $\bar{4}_6$	$\bar{4}_6$	$6r_4 + 1$ $r_4 = 0,1,2,3,...$	*odd* and *even*
	$12n(3n-1)+1$ $\bar{4}_6$	$\bar{2}_6$	$6r_2 - 1$ $r_2 = 0,1,2,3,...$	*odd* and *even*
	$4(1+3n)(2+3n)+1$ $\bar{6}_6$	$\bar{6}_6$	$6r_6 + 3$ $r_6 = 0,1,2,3,...$ $3 \mid N$	*odd* and *even*

Table 1: Square functions for the odd integers

$\underline{Z_6}$

Integers in this ring are given by $(6r_i + (i-3))$ (Table 2). The even integers occur in Classes $\bar{1}_6, \bar{3}_6, \bar{5}_6$. All integers in $\bar{3}_6$ have $3|N$, and there are no even powers in Class $\bar{5}_6$. The odd integers occur in Classes $\bar{2}_6, \bar{4}_6, \bar{6}_6$. All integers in $\bar{6}_6$ have $3|N$, while there are no even powers in Class $\bar{2}_6$.

Class → Row ↓	$\bar{1}_6$	$\bar{2}_6$	$\bar{3}_6$	$\bar{4}_6$	$\bar{5}_6$	$\bar{6}_6$
0	−2	−1	0	1	2	3
1	4	5	6	7	8	9
2	10	11	12	13	14	15
3	16	17	18	19	20	21
4	22	23	24	25	26	27
5	28	29	30	31	32	33

Table 2: Integers in Z_6

The classification of the various integers in these rings allows a more detailed analysis of infinite series since the contributions of the different classes can be readily assessed. Some examples which cover various infinite series are given in the next section.

3. Infinite Series

Table 3 lists some convergent infinite series. The aim here is to interpret these in terms of integer structure with the modular rings Z_4 and Z_6.

Type	**Series**	**Value**
A	$1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + \ldots$	$\pi^2/6 = \zeta(2)$
B	$1 - 1/2 + 1/4 - 1/5 + 1/7 - 1/8 + 1/10 - \ldots$	$\pi/3\sqrt{3}$
C	$(1/1^2 \times 1/2^2) + (1/3^2 \times 1/4^2) + (1/5^2 \times 1/6^2) + (1/7^2 \times 1/8^2) + \ldots$	$S(Z_4), S(Z_6)$
D	$\frac{8 \times 80 \times 224 \times 440 \times \ldots}{9 \times 81 \times 225 \times 441 \times \ldots}$	$\sqrt{3}/2$
E	$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + \ldots$	1
F	$\frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 8 \times \ldots}{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \times 9 \times \ldots}$	$\pi/2$
G	$1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + 1/13 - \ldots$	$\pi/4$
H	$1/(1 \times 2^1) - 1/(3 \times 2^3) + 1/(5 \times 2^5) - 1/(7 \times 2^7) + \ldots$ $+ 1/(1 \times 3^1) - 1/(3 \times 3^3) + 1/(5 \times 3^5) - 1/(7 \times 3^7) + \ldots + \ldots$	$\pi/4$
I	$4\{1/(1 \times 5^1) - 1/(3 \times 5^3) + 1/(5 \times 5^5) - 1/(7 \times 5^7) + \ldots\}$ $- \{1/(1 \times 239^1) - 1/(3 \times 239^3) + 1/(5 \times 239^5) - 1/(7 \times 239^7) + \ldots\} + \ldots$	$\pi/4$
J	$1/(1 \times 3) + 1/(5 \times 7) + 1/(9 \times 11) + 1/(13 \times 15) + \ldots$	$\pi/8$
K	$1/2 + 1/12 + 1/30 + 1/56 + 1/90 + \ldots$	$\ln 2$
L	$3 \times \frac{36}{35} \times \frac{144}{143} \times \frac{324}{323} \times \frac{576}{575} \times \ldots$	π
M	$1/(2 \times 3) + 1/(3 \times 4) + 1/(4 \times 5) + 1/(5 \times 6) + 1/(6 \times 7) + \ldots$	1/2

Table 3: Some convergent infinite series

Series A

Using Z_4 we can distinguish among the sums for the different classes of integers.

In $\bar{0}_4$, the integers are represented by $N = 4r_0$, so that $N^2 = 16r_0^2$, and the sum becomes

$$\begin{aligned} S(\bar{0}_4) &= \frac{1}{16}\sum_{r_0=1}^{\infty}\frac{1}{r_0^2} \\ &= \frac{1}{16}\zeta(2) \\ &= \frac{1}{6}\left(\frac{\pi}{4}\right)^2. \end{aligned}$$

The result shows that integers in this class are linked to the Euler and Machin Series (H and I respectively in Table 3).

In $\bar{2}_4$, $N = 4r_2 + 2$, and $N^2 = 4(2r_2 + 1)^2$, so that the sum becomes

$$S(\bar{2}_4) = \frac{1}{4}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^2}$$

which is one quarter the sum of the corresponding odd integers in Classes $\bar{1}_4$ and $\bar{3}_4$.

In $\bar{1}_4$, $N = 4r_1 + 1$ and $N^2 = 4R_1 + 1$ and the squares have three functions (Table 1). For $\bar{1}_4$ these are

$$\begin{aligned} &12n(3n+1)+1,\ (n = 0,2,4,6,...), \\ &12n(3n-1)+1,\ (n = 1,3,5,7,...), \\ &4(3n+1)(3n+2)+1,\ 3 \mid N,\ (n = 1,3,5,7,...). \end{aligned}$$

Hence the sum is

$$S(\bar{1}_4) = \sum_{n=0}^{\infty}\frac{1}{24n(6n+1)+1} + \frac{1}{24(2n+1)(3n+1)+1} + \frac{1}{8(3n+2)(6n+5)+1}.$$

Of course, the simple sum

$$S(\bar{1}_4) = \sum_{r=0}^{\infty}\frac{1}{(4r+1)^2}$$

also applies, but this fails to discriminate among the different types of integers within this class (Table 1).

In $\bar{3}_4$, $N = 4r_3 + 3$, but $N^2 = 4R_1 + 1$, as there are no even powers in this class. As in $\bar{1}_4$, the squares follow the three functions in Table 1 with reversed parity for n.

$$S(\bar{3}_4) = \sum_{n=0}^{\infty} \frac{1}{12(2n-1)(6n-2)+1} + \frac{1}{24(n+1)(6n+5)+1} + \frac{1}{8(6n+1)(3n+1)+1}.$$

As for class $\bar{1}_4$, there is a simple but restrictive sum:

$$S(\bar{3}_4) = \sum_{r=0}^{\infty} \frac{1}{(4r+3)^2}.$$

As can be seen from Table 1, the class structure in Z_6 is simpler than in Z_4, at least for the squares, so that only one function applies for each of the three classes of odd integers.

Series B

This sum obviously excludes the integers N such that $3|N$. Hence , the modular ring Z_6 is the most appropriate for any analysis, with integers of type $(6r+3) \in \bar{6}_6$ and $6r_3 \in \bar{3}_6$ being excluded and with $(1/N_i)$ following the pattern $\bar{1}_6\bar{2}_6\bar{4}_6\bar{5}_6$ for N; that is, each set of consecutive four-sum-components may be represented by

$$\frac{1}{(6r_1-2)} - \frac{1}{6r_2-1} + \frac{1}{6r_4+1} - \frac{1}{6r_5+2}$$

and, since $r_1 = r_2 = r_4 = r_5$, the series may be expressed as

$$\frac{\pi}{3\sqrt{3}} = \frac{1}{2} + \sum_{r=1}^{\infty} \frac{18r^2+1}{(9r^2-1)(36r^2-1)}.$$

Series C

For Z_4 we have the Class sequence $\left\{\left(\frac{1}{\bar{1}_4^2 \times \bar{2}_4^2}\right) + \left(\frac{1}{\bar{3}_4^2 \times \bar{0}_4^2}\right)\right\}$ repeated with $r = r_2 = r_3 = r_4$ and $r_0 = r+1$, so that

$$S(Z_4) = \sum_{r=0}^{\infty} \frac{1}{(4r+1)^2 \times (4r+2)^2} + \frac{1}{(4r+3)^2 \times (4r+4)^2}.$$

For Z_6 we have the Class sequence $\left\{\left(\frac{1}{\bar{4}_6^2 \times \bar{5}_6^2}\right) + \left(\frac{1}{\bar{6}_6^2 \times \bar{1}_6^2}\right) + \left(\frac{1}{\bar{2}_6^2 \times \bar{3}_6^2}\right)\right\}$ repeated with $r = r_4 = r_5 = r_6$ and $r_1 = r_2 = r_3 = r+1$, so that

$$S(Z_6) = \sum_{r=0}^{\infty} \frac{1}{(6r+1)^2 \times (6r+2)^2} + \frac{1}{(6r+3)^2 \times (6r+4)^2} + \frac{1}{(6r+5)^2 \times (6r+6)^2}.$$

Series D

For Z_4, this becomes

$$\frac{\sqrt{3}}{2} = \prod_{n=0}^{\infty} \frac{4(3n+1)(3n+2)}{4(3n+1)(3n+2)+1}$$

since $3|N$ and $2 \nmid N$.

For Z_6, this becomes

$$\frac{\sqrt{3}}{2} = \prod_{r=0}^{\infty} \frac{(6r+3)^2 - 1}{(6r+3)^2}$$

since $\overline{6}_6$ only is involved.

Series E

Here $\frac{1}{N}, N \in \overline{0}_4 \subset Z_4$, has each successive row double the previous one, and so the sum is

$$S'(\overline{0}_4) = \frac{1}{2} + \sum_{j=0}^{\infty} \frac{1}{2^{j+2}}.$$

Series F

In terms of the Z_4 class structure

$$\frac{\pi}{2} = \frac{\overline{2}_4\overline{2}_4\overline{0}_4\overline{0}_4\overline{2}_4\overline{2}_4\overline{0}_4\overline{0}_4}{\overline{1}_4\overline{3}_4\overline{3}_4\overline{1}_4\overline{1}_4\overline{3}_4\overline{3}_4\overline{1}_4},$$

so that

$$\frac{\pi}{2} = \prod_{r=0}^{\aleph} \frac{(4r+2)^2(4r+4)^2}{(4r+1)^2(4r+3)^2(4r+5)^2}.$$

For Z_6, the structure is

$$\frac{\overline{5}_6\overline{5}_6\overline{1}_6\overline{1}_6\overline{3}_6\overline{3}_6}{\overline{4}_6\overline{6}_6\overline{6}_6\overline{2}_6\overline{2}_6\overline{4}_6}$$

with $r_4 = r_5 = r_6 = r;\ r_1 = r_2 = r_3 = r_4 = r+1.$

Series G

For Z_4 this sum becomes

$$S(Z_4) = \frac{\pi}{4}$$

$$= 2\sum_{r=0}^{\infty} \frac{1}{(4r+1)(4r+3)} \quad .$$

$$= 1 - 2\sum_{r=0}^{\infty} \frac{1}{(4r+3)(4r+5)}.$$

<u>Series H and I</u>

These may be expressed as $\left(6S(\bar{0}_4)\right)^{1/2}$. Readers might like to express these series in terms of the rows $r \in \{Z_4, Z_6\}$.

<u>Series J</u>

This series is actually Series G. If we use Z_6 we get

$$\frac{\pi}{8} = \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{\left(36(2r-1)^2 - 1\right)} + \frac{1}{(12r-1)(12r-3)} + \frac{1}{(12r+3)(12r+1)}.$$

<u>Series K</u>

This series is related to Series C. Both are specific cases of the more general

$$S = \frac{1}{1^n \times 2^n} + \frac{1}{3^n \times 4^n} + \frac{1}{5^n \times 6^n} + \ldots \; .$$

The class structure in Z_4 here has the form

$$\frac{1}{\bar{1}_4 \times \bar{2}_4} + \frac{1}{\bar{3}_4 \times \bar{0}_4}$$

which is repeated with changing row, and so

$$\ln 2 = \sum_{r=0}^{\infty} \frac{1}{(4r+1)(4r+2)} + \frac{1}{(4r+3)(4r+4)}.$$

<u>Series L</u>

Here all the elements of the numerator belong to $\bar{3}_6 \in Z_6$. Hence

$$\pi = 3\prod_{r=0}^{\infty} \frac{36r^2}{36r^2 - 1}.$$

This can be compared with Series F and its more complicated structure.

Series M

The class structure here is

$$\frac{1}{\bar{2}_4\bar{3}_4}+\frac{1}{\bar{3}_4\bar{0}_4}+\frac{1}{\bar{0}_4\bar{1}_4}+\frac{1}{\bar{1}_4\bar{2}_4}$$

with $r_0 = r_1 = r_2' = r,\ r_2 = r_3 = r-1;$

$$\frac{1}{2}=\sum_{r=1}^{\infty}\frac{1}{4r-1}\left(\frac{1}{4(r-1)+2}+\frac{1}{4r}\right)+\frac{1}{4r+1}\left(\frac{1}{4r+2}+\frac{1}{4r}\right)$$

$$=\sum_{r=1}^{\infty}\frac{1}{4r^2-1}$$

which can be compared with E.

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