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THE BIRTHDAY INEQUALITY

Krassimir T. Atanassov
CLBME - Bulg. Academy of Sci., P.O.Box 12, Sofia-1113, Bulgaria,
e-mail: krat@bas.bg

To my daughter Vassia
for her birthday

We shall prove the following simple, but interesting

Theorem. Let $a_1, a_2, ..., a_n > 1$ be real numbers. Let $k \geq 1$ be an arbitrary natural number and let for every natural number m: $a_{n+m} = a_m$. Then

$$\sum_{i=1}^{n} a_i . \sum_{i=1}^{n} \frac{\log_{a_{i+1}} a_i}{a_i + a_{i+1} + ... + a_{i+k-1}} \geq \frac{n^2}{k}. \qquad (*)$$

Proof. Obviously, for given real numbers $a_1, a_2, ..., a_n > 1$:

$$\prod_{i=1}^{n} \log_{a_{i+1}} a_i = 1.$$

Then, using twice the well-known inequality

$$\sum_{i=1}^{n} x_i \geq n . \sqrt[n]{\prod_{i=1}^{n} x_i}$$

for the positive real numbers $x_1, ..., x_n$ we obtain

$$\sum_{i=1}^{n} a_i . \sum_{i=1}^{n} \frac{\log_{a_{i+1}} a_i}{a_i + a_{i+1} + ... + a_{i+k-1}}$$

$$\geq \sum_{i=1}^{n} a_i . n . \sqrt[n]{\frac{\prod_{i=1}^{n} \log_{a_{i+1}} a_i}{\prod_{i=1}^{n} (a_i + a_{i+1} + ... + a_{i+k-1})}}$$

$$= \sum_{i=1}^{n} a_i . \frac{n}{\sqrt[n]{\prod_{i=1}^{n} (a_i + a_{i+1} + ... + a_{i+k-1})}}$$

$$\geq \sum_{i=1}^{n} a_i . \frac{n^2}{\sum_{i=1}^{n}(a_i + a_{i+1} + \ldots + a_{i+k-1})}$$

$$= \sum_{i=1}^{n} a_i . \frac{n^2}{k. \sum_{i=1}^{n} a_i}$$

$$= \frac{n^2}{k},$$

i.e. (*) is valid.

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