

NOTE ON SOME IDENTITIES RELATED TO BINOMIAL COEFFICIENTS

Mladen V. Vassilev – Missana

5, V. Hugo Str., Sofia-1124, Bulgaria, e-mail: *missana@abv.bg*

The aim of the paper is the establishment of the following identities:

$$L^n = \sum_{k=0}^{\lfloor \frac{(L-1).n}{L} \rfloor} (-1)^k \binom{n}{k} \binom{L.(n-k)}{n}; \tag{1}$$

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} (-1)^k \binom{n}{k} \binom{3(n-k)}{n} = \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^n \binom{n}{k} \binom{k}{n-k} .3^k, \tag{2}$$

where $[x]$ denotes the integer part of x ; L and n are natural numbers.

Let $\{.\}$ be any expansion involving the variable x . Then by $coef_{x^s}\{.\}$ we shall denote the coefficient preceding x^s in this expansion.

Proof of (1): Using Newton's binomial formula we find

$$\begin{aligned} ((1+x)^L - x^L)^n &= \sum_{k \geq 0} (-1)^k \binom{n}{k} x^{L.k} (1+x)^{L.(n-k)} \\ &= \sum_{k \geq 0} (-1)^k \binom{n}{k} x^{L.k} \sum_{j \geq 0} \binom{L.(n-k)}{j} x^j \\ &= \sum_{k \geq 0} (-1)^k \binom{n}{k} \cdot \sum_{j \geq 0} \binom{L.(n-k)}{j} x^{j+Lk} \end{aligned}$$

Therefore,

$$coef_{x^{(L-1).n}} \{((1+x)^L - x^L)^n\} = \sum_{k \geq 0} (-1)^k \binom{n}{k} \cdot \binom{L.(n-k)}{(L-1).n - Lk}$$

(since $j + Lk = (L-1).n$)

$$= \sum_{k \geq 0} (-1)^k \binom{n}{k} \cdot \binom{L.(n-k)}{n}$$

(since $j \geq 0$ and $j = (L-1).n - L.k$ imply that $(L-1).n - L.k \geq 0$, i.e., $k \leq \lfloor \frac{(L-1).n}{L} \rfloor$)

$$= \sum_{k=0}^{\lfloor \frac{(L-1).n}{L} \rfloor} (-1)^k \binom{n}{k} \cdot \binom{L.(n-k)}{n}.$$

Hence

$$\text{coe} f_{x^{(L-1)n}} \{((1+x)^L - x^L)^n\} = \sum_{k=0}^{\lfloor \frac{(L-1)n}{L} \rfloor} (-1)^k \binom{n}{k} \cdot \binom{L(n-k)}{n}. \quad (3)$$

On the other hand, we have

$$((1+x)^L - x^L)^n = (1 + \binom{L}{1}x + \binom{L}{2}x^2 + \dots + \binom{L}{L-1}x^{L-1})^n.$$

Hence

$$\text{coe} f_{x^{(L-1)n}} \{((1+x)^L - x^L)^n\} = \binom{L}{L-1}^n = L^n \quad (4).$$

Now, (3) and (4) prove (1).

When $L = 2$, (1) yields the identity

$$2^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2(n-k)}{n}. \quad (5)$$

The last identity is contained in [1], but in the form

$$2^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \binom{2(n-k)}{n-k} \quad (6)$$

since

$$\binom{n}{n-k} \binom{2(n-k)}{n} = \binom{n-k}{k} \binom{2(n-k)}{n-k}.$$

The same way as in the proof of (1) we may establish that:

$$\text{coe} f_{x^{Rn}} \{((1+x)^L - x^L)^n\} = \sum_{k=0}^{\lfloor \frac{Rn}{L} \rfloor} (-1)^k \binom{n}{k} \cdot \binom{L(n-k)}{(L-R)n}, \quad (7)$$

$$\text{coe} f_{x^{Rn}} \{((1+x)^L - x^L)^n\} = \text{coe} f_{x^{Rn}} \{1 + \binom{L}{1}x + \binom{L}{2}x^2 + \dots + \binom{L}{L-1}x^{L-1}\}^n, \quad (8)$$

where R is a natural number, such that $R \leq L$.

Proof of (2): Using Newton's binomial formula we find

$$((1+x)^L - x^L)^n = \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{s=0}^{L(n-k)} \binom{L(n-k)}{s} x^{s+Lk}.$$

Hence,

$$\text{coe} f_{x^n} \{((1+x)^L - x^L)^n\} = \sum_{k=0}^{\lfloor \frac{n}{L} \rfloor} (-1)^k \binom{n}{k} \cdot \binom{L(n-k)}{n}, \quad (9)$$

since

$$\binom{L.(n-k)}{n-Lk} = \binom{L.(n-k)}{Ln-n} = \binom{L.(n-k)}{n}.$$

On the other hand we have

$$\text{coe}f_{x^n}\{((1+x)^L - x^L)^n\} = \text{coe}f_{x^n}\left\{\left(\sum_{k=0}^{L-1} \binom{L}{k} .x^k\right)^n\right\}. \quad (10)$$

Now, (9) and (10) imply

$$\sum_{k=0}^{\lfloor \frac{n}{L} \rfloor} (-1)^k \binom{n}{k} . \binom{L.(n-k)}{n} = \text{coe}f_{x^n}\left\{\left(\sum_{k=0}^{L-1} \binom{L}{k} .x^k\right)^n\right\}. \quad (11)$$

Let $L = 3$. Then:

$$\begin{aligned} \text{coe}f_{x^n}\left\{\left(\sum_{k=0}^{L-1} \binom{L}{k} .x^k\right)^n\right\} &= \text{coe}f_{x^n}\left\{\left(\sum_{k=0}^2 \binom{3}{k} .x^k\right)^n\right\} \\ &= \text{coe}f_{x^n}\{(3x^2 + 3x + 1)^n\} = \text{coe}f_{x^n}\{(3x^2 + (3x + 1))^n\} \\ &= \text{coe}f_{x^n}\left\{\sum_{k=0}^n \binom{n}{k} .3^{n-k} .x^{2(n-k)} .(3x + 1)^k\right\} \\ &= \text{coe}f_{x^n}\left\{\sum_{k=0}^n \binom{n}{k} .3^{n-k} .x^{2(n-k)} . \sum_{j=0}^k \binom{k}{j} .3^j .x^j\right\} \\ &= \text{coe}f_{x^n}\left\{\sum_{k=0}^n \binom{n}{k} .3^{n-k} . \sum_{j=0}^k \binom{k}{j} .3^j .x^{2(n-k)+j}\right\}. \end{aligned}$$

Now, (11) (the case $L = 3$) and the last equality yield

$$\begin{aligned} \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} (-1)^k \binom{n}{k} . \binom{3.(n-k)}{n} \\ = \text{coe}f_{x^n}\left\{\sum_{k=0}^n \binom{n}{k} .3^{n-k} . \sum_{j=0}^k \binom{k}{j} .3^j .x^{2(n-k)+j}\right\}. \end{aligned} \quad (12)$$

Let $2(n-k) + j = n$. Then $j = 2k - n$ and since $j \geq 0$, it is fulfilled $2k - n \geq 0$, i.e., $k \geq \lfloor \frac{n+1}{2} \rfloor$.

On the other hand we have $k \leq n$. Hence

$$\lfloor \frac{n+1}{2} \rfloor \leq k \leq n. \quad (13)$$

