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The sum-of-divisors minimum and maximum functions

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1. Let $f : \mathbb{N}^* \to \mathbb{N}$ be a given arithmetic function, and $A \subset \mathbb{N}^*$ a given set. The arithmetic function

$$F_f^A(n) = \min\{k \in A : \ n|f(k)\} \tag{1}$$

has been introduced in [7] and [6]. For $A = \mathbb{N}^*$, $f(k) = k!$ one obtains the Smarandache function; for $A = \mathbb{N}^*$, $A = P = \{2, 3, 5, \dots\}$ = set of all primes, one obtains a function

$$P(n) = \min\{k \in P : \ n|k!\} \tag{2}$$

For properties of this function, see [7], [6].

For $A = \{k^2 : \ k \in \mathbb{N}^*\}$ = set of perfect squares, and $f(k) = k!$ one obtains the function

$$Q(n) = \min\{m^2 \geq 1 : \ n|(m^2)!\}, \tag{3}$$

while for A = set of squarefree numbers ≥ 1, $f(k) = k!$ we get

$$Q_1(n) = \min\{m \geq 1 \text{ squarefree: } n|m!\} \tag{4}$$

For properties of $Q(n)$ and $Q_1(n)$, see [11].

The "dual" function of (1) has been defined by

$$G_g^A(n) = \max\{k \in A : \ g(k)|n\} \tag{5}$$

where $g : \mathbb{N}^* \to \mathbb{N}$ is a given function. Particularly for $A = \mathbb{N}^*$, $g(k) = k!$ one obtains the dual of the Smarandache function

$$S_*(n) = \max\{k \geq 1 : \ k!|n\} \tag{6}$$

For properties of this function, see [7], [6]. F. Luca [4], K. Atanassov [1] and M. Le [2] have proved in the affirmative a conjecture of the author stated in [7] and [6].

For $A = \mathbb{N}^*$, $f(k) = g(k) = \varphi(k)$ (where φ is Euler's totient) in (1), resp. (5) one obtains the Euler minimum, resp. maximum-functions, defined by

$$E(n) = \min\{k \geq 1 : \ n|\varphi(k)\}, \tag{7}$$

$$E_*(n) = \max\{k \geq 1 : \ \varphi(k)|n\} \tag{8}$$

For properties of these functions, see [5], [8].

When $A = \mathbb{N}^*$, $f(k) = d(k) =$ number of divisors of k, one has the divisor minimum function (see [7], [6], [9]):

$$D(n) = \min\{k \geq 1 : \ n|d(k)\} \tag{9}$$

It is interesting to note that the divisor maximum function (i.e. the "dual" of $D(n)$) given by

$$D_*(n) = \max\{k \geq 1 : \ d(k)|n\} \tag{10}$$

is not well-defined! Indeed, for any prime p we have $d(p^{n-1}) = n$, and p^{n-1} is unbounded as $p \to \infty$. When A is a finite set, however,

$$D_*^A(n) = \max\{k \in A : \ d(k)|n\} \tag{11}$$

does exist.

When $A = \mathbb{N}^*$, $f(k) = g(k) = S(k) = \min\{m \geq 1 : \ k|m!\}$ (Smarandache function) one obtains the Smarandache minimum and maximum functions, given by

$$S_{min}(n) = \min\{k \geq 1 : \ n|S(k)\}, \tag{12}$$

$$S_{max}(n) = \max\{k \geq 1 : \ S(k)|n\}. \tag{13}$$

These functions have been introduced and studied recently in [10].

2. Let $\sigma(n)$ be the sum of divisors of n. The function

$$\Sigma(n) = \min\{k \geq 1 : \ n|\sigma(k)\} \tag{14}$$

has been introduced in [7], [6] (denoted there by F_σ). Let k be a prime of the form $k = an - 1$, where $n \geq 1$ is given. By Dirichlet's theorem on arithmetical progressions, such a prime does exist. Then clearly $\sigma(k) = an$, so $n|\sigma(k)$, and $\Sigma(n)$ is well defined.

The dual of $\Sigma(n)$ is

$$\Sigma_*(n) = \max\{k \geq 1 : \ \sigma(k)|n\} \tag{15}$$

Since $\sigma(1) = 1|n$ and $\sigma(k) \geq k$, clearly $\Sigma_*(n) \leq n$, so this function is correctly defined.

The aim of this note is the initial study of these functions $\Sigma(n)$ and $\Sigma_*(n)$.

Some values of $\Sigma(n)$ are: $\Sigma(1) = 1$, $\Sigma(2) = 3$, $\Sigma(3) = 2$, $\Sigma(4) = 3$, $\Sigma(5) = 8$, $\Sigma(6) = 5$, $\Sigma(7) = 4$, $\Sigma(8) = 7$, $\Sigma(9) = 10$, $\Sigma(11) = 43$, $\Sigma(12) = 6$, $\Sigma(13) = 9$, $\Sigma(14) = 12$, $\Sigma(15) = 8$, $\Sigma(16) = 21$, $\Sigma(17) = 67$, $\Sigma(18) = 10$, $\Sigma(19) = 37$, $\Sigma(20) = 19$, $\Sigma(21) = 20$, $\Sigma(22) = 43$, $\Sigma(23) = 137$, $\Sigma(24) = 14$, $\Sigma(25) = 149$, $\Sigma(26) = 45$, $\Sigma(27) = 34$, $\Sigma(28) = 12$, $\Sigma_*(1) = 1$, $\Sigma_*(2) = 1$, $\Sigma_*(3) = 2$, $\Sigma_*(4) = 3$, $\Sigma_*(5) = 1$, $\Sigma_*(6) = 5$, $\Sigma_*(7) = 4$, $\Sigma_*(8) = 7$, $\Sigma_*(9) = 2$, $\Sigma_*(10) = 1$, $\Sigma_*(11) = 1$, $\Sigma_*(12) = 11$, $\Sigma_*(13) = 9$, $\Sigma_*(14) = 13$, $\Sigma_*(15) = 8$, $\Sigma_*(16) = 7$, $\Sigma_*(17) = 1$, $\Sigma_*(18) = 17$, $\Sigma_*(19) = 1$, $\Sigma_*(20) = 19$, $\Sigma_*(21) = 4$, $\Sigma_*(22) = 1$, $\Sigma_*(23) = 1$, $\Sigma_*(24) = 23$, $\Sigma_*(25) = 1$, $\Sigma_*(26) = 9$, $\Sigma_*(27) = 2$, $\Sigma_*(28) = 12$.

3. The first theoretical result gives informations on values of these functions at $n = p + 1$, where p is a prime:

Theorem 1. *If p is a prime, then*

$$\Sigma(p + 1) \leq p \leq \Sigma_*(p + 1) \tag{16}$$

Proof. Since $(p + 1)|\sigma(p) = p + 1$, by definition (14) one can write $\Sigma(p + 1) \leq p$. Similarly, definition (15) gives (by $\sigma(p) = (p + 1)|(p + 1)$) $\Sigma_*(p + 1) \geq p$.

Remark. On the left side of (16) one can have equality, e.g. $\Sigma(3) = 2$, $\Sigma(6) = 5$, $\Sigma(8) = 7$. But the inequality can be strict, as $\Sigma(12) = 6 < 11$, $\Sigma(18) = 10 < 17$. For the right side of (16) however, one can prove the more precise result:

Theorem 2. *For all primes p, one has*

$$\Sigma_*(p + 1) = p \tag{17}$$

Proof. First we prove that for all $n \geq 2$ we have

$$\Sigma_*(n) \leq n - 1 \tag{18}$$

Indeed, since $\sigma(k)|n$, clearly we must have $\sigma(k) \leq n$. On the other hand, for all $k \geq 2$ we have $\sigma(k) \geq k + 1$ (with equality only for $k =$ prime), so $k \leq n - 1$, and this is true for all k, so (18) follows.

Let now $n = p + 1 \geq 3$ in (18). Then $\Sigma_*(p + 1) \leq p$, which combined with (16) implies relation (17).

Theorem 3. *Let p be a prime and suppose that*

$$(p + 1)|n \tag{19}$$

Then

$$\Sigma_*(n) \geq p \tag{20}$$

Proof. Indeed, by $\sigma(p) = (p+1)|n$, and definition (15), relation (20) follows. By letting $p = 2, 3, 5, 7, 11$ one gets:

Corollary. *If* $3|n$, *then* $\Sigma_*(n) \geq 2$. (21)

If $4|n$, *then* $\Sigma_*(n) \geq 3$. (22)

If $6|n$, *then* $\Sigma_*(n) \geq 5$. (23)

If $8|n$, *then* $\Sigma_*(n) \geq 7$. (24)

If $12|n$, *then* $\Sigma_*(n) \geq 11$. (25)

Remark. If $7|n$, then $\Sigma_*(n) \geq 4$. (26)

Indeed, $\sigma(4) = 7|n$.

If $15|n$, then $\Sigma_*(n) \geq 8$. (27)

Indeed, $\sigma(8) = 15|n$.

It is immediate that $\Sigma(n) = 1$ only for $n = 1$. On the other hand, there exist many integers m with $\Sigma_*(m) = 1$.

Theorem 4. *Let p be a prime such that*

$$p \notin \sigma(\mathbb{N}^*) \tag{28}$$

Then

$$\Sigma_*(p) = 1 \tag{29}$$

Proof. Remark that $\sigma(k)|p \Leftrightarrow \sigma(k) = 1$ or $\sigma(k) = p$. Now, if (28) is true, then the equation $\sigma(k) = p$ is impossible for all $k \geq 1$, so $\sigma(k) = 1$, i.e. $k = 1$, giving relation (29).

For example, $p = 17, 19, 23$ satisfy relation (28).

Theorem 5. *If for all $d > 1$, $d|n$ one has*

$$d \notin \sigma(\mathbb{N}^*), \tag{30}$$

then

$$\Sigma_*(n) = 1 \tag{31}$$

Proof. Let $d > 1$, $d|n$. If $d \notin \sigma(\mathbb{N}^*)$, then the equation $\sigma(k) = d$ is impossible. But then $\sigma(k)|n$ is also impossible for $\sigma(k) > 1$, yielding (31).

For example, $n = 10, 22, 25$ satisfy relation (30).

Theorem 6. *Let n be odd and suppose that $\Sigma_*(n) \neq 1, 2$. Then*

$$\Sigma_*(n) \leq \left(\frac{-1 + \sqrt{-3 + 4n}}{2} \right)^2 \tag{32}$$

Proof. We use the following well-known results:

Lemma 1. *$\sigma(k)$ is odd iff $k = m^2$ or $k = 2^\alpha m^2$, where $\alpha \geq 1$ and m is an odd integer.* (33)

Proof. Let $k = p_1^{\alpha_1} \dots p_r^{\alpha_r}$. Then

$$\sigma(k) = (1 + p_1 + \cdots + p_1^{\alpha_1}) \dots (1 + p_r + \cdots + p_r^{\alpha_r}).$$

If k is odd, the $\sigma(k)$ is odd if each term $1 + p_1 + \cdots + p_1^{\alpha_1}, \dots, 1 + p_r + \cdots + p_r^{\alpha_r}$ is odd, and since p_i $(1 = \overline{1, r})$ are all odd numbers, we must have $\alpha_1 =$ even, ..., $\alpha_r =$ even. This gives $k = m^2$, with $m =$ odd. When k is even, then $k = 2^\alpha p_1^{\alpha_1} \dots p_r^{\alpha_r}$, and since $\sigma(2^\alpha) = 2^\alpha + 1 =$ odd, by the same argument as above, $k = 2^\alpha m^2$, with $m =$ odd.

Lemma 2. *If k is composite, then*

$$\sigma(k) \geq k + \sqrt{k} + 1 \tag{34}$$

Proof. Write $k = ab$, where $1 < a \leq b < k$. Then $k \leq b^2$, so $b \geq \sqrt{k}$, implying $\sigma(k) \geq 1 + b + k \geq 1 + \sqrt{k} + k$, i.e. relation (34). When $k = p^2$, with p an odd prime, one has equality since $\sigma(p^2) = p^2 + p + 1$.

Now, if $\sigma(k)|n$ and n is odd, then clearly $\sigma(k)$ must be odd, too. Now, by (33) this is possible only when $k = m^2$ or $k = 2^\alpha m^2$, with $m \geq 1$ odd. If $m > 1$, then $k = m^2$ is composite, while if $m = 1$ in $k = 2^\alpha m^2$, then $k = 2^\alpha$ is prime only if $\alpha = 1$, i.e. if $k = 2$. Supposing $k \neq 1, 2$ then k is always composite, so $\sigma(k) \geq k + \sqrt{k} + 1$. Since $\sigma(1) \leq n$, we get $k + \sqrt{k} + 1 - n \leq 0$ so $\sqrt{k} \leq \dfrac{-1 + \sqrt{-3 + 4n}}{2}$, and this gives (32).

Remark. For example, by (26), for $7|n$, n odd, (32) is true.

Theorem 7. *If $n \geq 4$, then $\Sigma(n) \geq 3$. For all $n \geq 4$,*

$$\Sigma(n) > n^{2/3} \tag{35}$$

Proof. $\Sigma(n) = 1$ iff $n|1$, when $n = 1$. For $\Sigma(n) = 2$ we have $\sigma(2) = 3$ so $n|3 \Leftrightarrow n = 1, 3$. Thus for $n \geq 4$, we have $k = \Sigma(n) \geq 3$. Now, if $n|\sigma(k)$, then clearly $n \leq \sigma(k)$. Let $k \geq 3$.

Then, it is known (see [3]) that

$$\sigma(k) < k\sqrt{k} \tag{36}$$

By $n < k\sqrt{k} = k^{3/2}$, inequality (35) follows.

Corollary. *For all $m \geq 2$ (left side), and $m \geq 1$ (right side):*

$$(2^{m+1}-1)^{2/3} < \Sigma(2^{m+1}-1) \leq 2^m \tag{37}$$

Proof. $2^{m+1}-1 > 4$ for $m \geq 2$, and the left side is a consequence of (35). Now, the right side follows by $(2^{m+1}-1)|\sigma(2^m)$, since $\sigma(2^m) = 2^{m+1}-1$, and apply definition (14).

Theorem 8. *Let $f : [1,\infty) \to [1,\infty)$ be given by $f(x) = x + x\log x$. Then for all $n \geq 1$,*

$$\Sigma(n) \geq f^{-1}(n), \tag{38}$$

where f^{-1} is the inverse function of f.

Proof. $\sigma(n) = \sum_{d|n} d = \sum_{d|n} \frac{n}{d} = n\sum_{d|n}\frac{1}{d} \leq n\sum_{1\leq d\leq n}\frac{1}{d} \leq n(1+\log n)$ as it is well known that $1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \log n$ for all $n \geq 1$. Thus if $n|\sigma(k)$, then $n \leq \sigma(k) \leq f(k)$, so (38) follows. The function f is strictly increasing and continuous, so it is bijective, having an inverse function $f^{-1} : [1,\infty) \to [1,\infty)$.

Remark. The inequality $f(x) < x\sqrt{x}$, i.e. $\log x < \sqrt{x}-1$ is true for x sufficiently large (e.g. $x \geq e^3$). Indeed, let $g(x) = \sqrt{x} - \lg x - 1$, when $g(e^3) = e^{3/2} - 4 > 0$ by $e^3 \approx 19.6 > 4^2 = 16$, and $g'(x) = \frac{\sqrt{x}-2}{2x} > 0$ for $x > 4$. So $g(x) \geq g(e^3) > 0$ for $x \geq e^3$. Thus $x + x\lg x < x\sqrt{x}$. By putting $x = n^{2/3}$ we get $f(n^{2/3}) < n$, i.e. for $n^{2/3} \geq e^3$ ($m \geq e^{9/2}$) we get:

$$f^{-1}(n) > n^{2/3} \text{ for } n \geq e^{9/2} \tag{39}$$

which improves, by (38), inequality (35).

For values of $\Sigma(n)$ and $\Sigma_*(n)$ at primes $n = p$ the following is true:

Theorem 9. *For all primes $p \geq 5$,*

$$1 \leq \Sigma_*(p) \leq p-2 \tag{40}$$

and

$$\Sigma_*(p) \leq \left(\frac{-1+\sqrt{-3+4p}}{2}\right)^2 \tag{41}$$

Proof. The inequality $\Sigma_*(n) \geq 1$ is true for all n (but remains an Open Problem the determination of all n with equality). Now, remark that $\sigma(k)|p$ iff $\sigma(k) = 1$ or $\sigma(k) = p$. If $\sigma(k) > 1$, then by $\sigma(k) \geq k + 1$ we get $k \leq p - 1$. But we cannot have equality, since then $k = q =$ prime, when $\sigma(q) = q + 1 = p \geq 5$ and this is impossible, since $q + 1$ is even for $q \geq 3$, while for $q = 2$, $q + 1 = 3 < 5$. Thus $k \leq p - 2$, so (40) follows. By applying the inequality $\sigma(k) \geq k + \sqrt{k} + 1$ (see (34)) then one arrives at (41), which is sharp, since e.g. $\Sigma_*(7) = 4 \leq 4$.

Theorem 10. *For all Mersenne primes p one has*

$$\Sigma(p) \leq \frac{p+1}{2} \tag{42}$$

Proof. This follows from the right side of (37), by remarking that when $p = 2^{m+1} - 1$ is a prime, by $\Sigma(2^{m+1} - 1) \leq 2^m = \dfrac{p+1}{2}$ we get (42).

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