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REMARKS ON THE 46-TH SMARANDACHE'S PROBLEM

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The 46-th Smarandache's problem from [3] is the following:

Smarandache's prime additive complements:

$$1,0,0,1,0,1,0,3,2,1,0,1,0,3,2,1,0,1,0,3,2,1,0,1,0,5,4,3,2,1,$$

$$0,1,0,5,4,3,2,1,0,3,2,1,0,5,4,3,2,1,0,...$$

(For each n to find the smallest k such that $n+k$ is prime.)

Remark: Smarandache asked if it is possible to get as large as we want but finite decreasing $k,\ k-1,\ k-2,...,2,1,0$ (odd k) sequence included in the previous sequence – i.e., for any even integer are there two primes those difference is equal to it? He conjectured the answer is negative.

Obviously, the members of the above sequence are differences between first prime number that is greater or equal to the current natural number n and the same n. It is well-known that the number of primes smaller than or equal to n is $\pi(n)$. Therefore, the prime number smaller than or equal to n is $p_{\pi(n)}$.

In [2] three new formulae for $\pi(n)$ and a new formula for the n-th prime number p_n are introduced. Now we shall introduce another – simpler formula for $\pi(n)$ and p_n.

Let us define functions sg and $\overline{sg}$ by:

$$sg(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases},$$

$$\overline{sg}(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases},$$

where x is a real number.

For the natural number

$$1 < n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, α_1, α_2, ... $\alpha_k \geq 1$ are natural numbers and $p_1, p_2, ..., p_k$ are different prime numbers, let us define (see [1]) the arithmetical function η by:

$$\eta(n) = \sum_{i=1}^{n} \alpha_i.p_i.$$

Then the following equality holds for every natural number $n \geq 2$:

$$\pi(n) = \sum_{k=2}^{n} \overline{sg}(k - \eta(k)),$$

Really, for every natural number k, such that $k \leq n$, if k is prime, then $\overline{sg}(k - \eta(k)) = 1$. On the other hand, if k is not prime, then $k - \eta(k) > 0$, i.e., $\overline{sg}(k - \eta(k)) = 0$. Therefore, the sum is equal to $\pi(n)$.

Of course, $\pi(0) = 0$ and $\pi(1) = 0$.

For the so constructed formula for $\pi(n)$, by analogy with [2] we can prove that for every natural number n:

$$p_n = \sum_{i=0}^{2^n} sg(n - \pi(i)).$$

Returning to the Smarandache's problem, we shall note that the prime number that is greater than or equal to n is the next prime number, i.e., $p_{\pi(n)+1}$. Finally, the n-th member of the Smarandache's sequence will be equal to

$$\begin{cases} p_{\pi(n)+1} - n, & \text{if } n \text{ is not a prime number} \\ 0, & \text{otherwise} \end{cases}$$

We shall note that in [2] the following new formula p_n for every natural number n is given:

$$p_n = \sum_{i=0}^{\theta(n)} sg(n - \pi(i)),$$

where $\theta(n) = [\frac{n^2 + 3n + 4}{4}]$ (for $\theta(n)$ see [5]).

Let a_n denote the n-th term of the Smarandache's sequence. Then, we propose a way for obtaining an explicit formula for a_n ($n = 1, 2, 3, ...$). Extending the below results, we give an answer to the Smarandache's question from his own Remark. At the end, we propose a generalization of Problem 46 and present a proof of an assertion related to Smarandache's conjecture for Problem 46.

Proposition 1. a_n admits the representation

$$a_n = p_{\pi(n-1)+1} - n, \tag{1}$$

where $n = 1, 2, 3, ...$

The proof is a matter of direct check.

It is clear that (1) gives an explicit representation of a_n since several explicit formulae for $\pi(k)$ and p_k are known (see, e.g. [4]).

Let us define

$$n(m) = m! + 2.$$

Then all numbers

$$n(m), n(m) + 1, n(m) + 2, ..., n(m) + m - 2$$

are composite. Hence

$$a_{n(m)} \geq m - 1.$$

This proves Smarandache's conjecture, since m may grow up to infinity. Therefore $\{a_n\}_{n=1}^{\infty}$ is unbounded sequence.

Now, we shall generalize Problem 46.

Let

$$\mathbf{c} \equiv c_1, c_2, c_3, ...$$

be a strictly increasing sequence of positive integers.

Definition. Sequence

$$\mathbf{b} \equiv b_1, b_2, b_3, \dots$$

is called c-additive complement of $\mathbf{c}$ if and only if b_n is the smallest non-negative integer, such that $n + b_n$ is a term of $\mathbf{c}$.

The following assertion generalizes Proposition 1.

Proposition 2. b_n admits the representation

$$b_n = c_{\pi_c(n-1)+1} - n, \tag{2}$$

where $n = 1, 2, 3, \dots$, $\pi_c(n)$ is the counting function of $\mathbf{c}$, i.e., $\pi_c(n)$ equals to the quantity of $c_m, m = 1, 2, 3, \dots$, such that $c_m \leq n$.

We omit the proof since it is again a matter of direct check.

Let

$$d_n \equiv c_{n+1} - c_n \ (n = 1, 2, 3, \dots).$$

The following assertion is related to Smarandache's conjecture from Problem 46.

Proposition 3. If $\{d_n\}_{n=1}^{\infty}$ is unbounded sequence, then $\{b_n\}_{n=1}^{\infty}$ is unbounded sequence, too.

Proof. Let $\{d_n\}_{n=1}^{\infty}$ be unbounded sequence. Then there exists a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$, such that sequence $\{d_{n_k}\}_{k=1}^{\infty}$ is strictly increasing, too. Hence $\{d_n\}_{n=1}^{\infty}$ is unbounded sequence, since it contains a strictly increasing sequence of positive integers.

Open Problem. Formulate necessary conditions for the sequence $\{b_n\}_{n=1}^{\infty}$ to be unbounded.

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