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RELATIONSHIP BETWEEN A NATURAL NUMBER AND ITS DIGITAL PRODUCT

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Abstract. For the integer number $n \geq 1$ written in the basis $b \geq 2$, we denote by $s_b(n)$, $p_b(n)$ and $d_b(n)$ the sum, the product and the number of its digits, respectively. There are a series of more recent or older papers concerning the relationship between these numbers. For instance, one proves in [2] that, if $b \geq 3$, then for infinitely many numbers m there exists n with $d_b(n) = m$ and $s_b(n) = m$, and this property fails also for infinitely many values of m.

1. An inequality involving $p(n)$

Starting from the obvious inequality $p_b(n) \leq n$, we are going to prove a stronger result, namely:

Theorem 1. *For every $n > b$ we have*

$$n^{1/k} - p_b(n)^{1/k} > 1 \tag{1}$$

where k is an integer such that $1 \leq k < d_b(n)$.

Proof. Let $d = d_b(n) \geq 2$ and $n = \sum_{i=0}^{d-1} a_i b^i$. Inequality (1) is obvious when $p_b(n) = 0$. If $p_b(n) > 0$, then $n > a_{d-1} b^{d-1}$ and $p_b(n) = a_0 a_1 \cdots a_{d-1} \leq a_{d-1}(b-1)^{d-1}$, whence

$$p_b(n) < \left(\frac{b-1}{b}\right)^{d-1} n. \tag{2}$$

It then follows that $p_b(n)^{1/(d-1)} \leq \frac{b-1}{b} \cdot n^{1/(d-1)}$, hence

$$n^{1/(d-1)} - p_b(n)^{1/(d-1)} > n^{1/(d-1)} \left(1 - \frac{b-1}{b}\right) = \frac{n^{1/(d-1)}}{b-1}$$

$$\geq \frac{(a_{d-1} b^{d-1})^{1/(d-1)}}{b} = a_{d-1}^{1/(d-1)} \geq 1.$$

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Inequality (1) is thus proved for $k = d - 1$.

Note that, for $a > b \geq 1$ and $x > 0$, the function f defined by $f(x) = a^x - b^x$ is increasing since $f'(x) = a^x \log a - b^x \log b > 0$. For $k \leq d - 1$ we have $1/k \geq 1/(d-1)$, hence

$$n^{1/k} - p_b(n)^{1/k} \geq n^{1/(d-1)} - p_b(n)^{1/(d-1)} > 1,$$

and the proof ends. □ □

Remark 1. For $n = b^d - 1$ we have $d_b(n) = d$ and

$$n^{1/d} - p_b(n)^{1/d} = (b^d - 1)^{1/d} - (b-1)^{d/d} < 1$$

since $(b^d - 1)^{1/d} < b$. Hence inequality (1) doesn't hold for $k = d_b(n)$.

We now rise the problem to determine the limit points of the sequence $(x_n)_{n \geq 1}$, where $x_n = n^{1/d} - p_b(n)^{1/d}$. The answer is given by

Theorem 2. *The set of all limit points of the sequence* $(x_n)_{n\geq 1}$ *is* $[1, b-1] \cup \{b\}$.

Proof. If n has d digits, that is $b^{d-1} \leq n \leq b^d - 1$, then $b^{1-1/d} \leq n^{1/d} < b$. For $n \to \infty$ it follows that $d \to \infty$, hence $\lim_{n\to\infty} n^{1/d} = b$.

It remains to estimate $p_b(n)^{1/d}$. If at least one digit of n equals 0, then $p_b(n) = 0$, hence $x_n = n^{1/d}$, and thus b is a limit point of the sequence $(x_n)_{n\geq 1}$.

If no digit of n equals 0, then $(b-1)^d \geq p_b(n) \geq 1$. Let $x \in [1, b-1]$, $b \geq 3$, and consider the numbers n having k digits equal to $b-1$ and $d-k$ digits equal to 1. It follows that $p_b(n)^{1/d} = (b-1)^{k/d}$. For $k = [d \log_{b-1} x]$ we have $k \leq d$ and, since $\lim_{d\to\infty} [d \log_{b-1} x]/d = \log_{b-1} x$, we get $\lim_{n\to\infty} p(n)^{1/d} = (b-1)^{\log_{b-1} x} = x$, and we obtain the desired conclusion.

Now let $b = 2$. If $p_2(n) = 0$, then we get the limit point 2. If $p_2(n) > 0$, then all the digits of n are equal to 1, hence $p_2(n_= 1,$, and we get the limit point 1. □ □

2. Divisibility problems

Since $n \geq p_b(n)$, it is natural to attempt determining the situations when n is divisible by $p_b(n)$. Such an attempt turns out to be quite difficult. Nevertheless, it is clear that for the numbers $n = 11\ldots1$ we have $p_b(n) = 1$ hence n is divisible by $p_b(n)$. Thus there exist infinitely many numbers n having this latter property.

We will solve the problem in some special cases. Let us denote $s_{10}(n) = s(n)$ and $p_{10}(n) = p(n)$. We have:

Theorem 3. *If $n = 2p(n)$ then $n = 36$.*
If $n = 3p(n)$ then we have either $n = 15$ or $n = 24$..

Proof. We have clearly $n > 10$ and $p(n) > 0$. According to (2) we have

$$p(n) < \left(\frac{9}{10}\right)^{d-1} n. \tag{3}$$

If $n = 2p(n)$, then $(10/9)^{d-1} < 2$, hence $d \leq 7$. It is easy to check that $n = 36$ is the only solution to the equation $n = 2p(n)$ for $n < 10^7$.

For the relation $n = 3p(n)$ we get the necessary condition $(10/9)^{d-1} < 3$, whence $d \leq 11$. The checking carried out for $n < 10^7$ leads to $n = 15$ and $n = 24$. It remains to prove that there is no solution for $d = 8, 9, 10, 11$. To this end, we first change the indices of $a_0, a_1, \ldots, a_{d-2}$ such that

$$x_1 \leq x_2 \leq \cdots \leq x_{d-1}.$$

If $x_4 < 9$, then $a_{d-1}10^{d-1} < 3a_{d-1}8^4 9^{d-5}$, whence we get the contradiction $2.09\ldots < (10/9)^{d-1} < 3 \cdot 8^4/9^4 = 1.87\ldots$. Hence $x_4 = 9$. Since $n \vdots 9$,, we get $s(n) \vdots 9$, hence $a_{d-1} + x_1 + x_2 + x_3 + (d-4)9 \vdots 9$. Hence $a_{d-1} + x_1 + x_2 + x_3 \vdots 9$. We have $a_{d-1}10^{d-1} < 3a_{d-1}9^{d-4}x_1x_2x_3$, whence $x_1x_2x_3 > (10/9)^{d-1} \cdot 243$. Since $d \geq 8$, we get $x_1x_2x_3 > 508$. There exist the following possible situations:

(a) $x_1 = x_2 = x_3 = 8$, $a_{d-1} = 3$;
(b) $x_1 = 7$, $x_2 = x_3 = 9$, $a_{d-1} = 2$;
(c) $x_1 = x_2 = 8$, $x_3 = 9$, $a_{d-1} = 2$;
(d) $x_1 = 8$, $x_2 = x_3 = 9$, $a_{d-1} = 1$;
(e) $x_1 = x_2 = x_3 = 9$, $a_{d-1} = 9$.

The obvious inequality $a_{d-1}10^{d-1} + x_1 10^{d-2} < 3a_{d-1}9^{d-4}x_1x_2x_3$ takes on the form $27(10/9)^{d-1} < a_{d-1}x_1x_2x_3/(10a_{d-1} + x_1)$, hence

$$a_{d-1}x_1x_2x_3 > 56.45(10a_{d-1} + x_1). \tag{4}$$

Inequality (4) holds in none of the cases (a), (b), (c), (d). On the other hand, in the case (e) doesn't hold the equality $n = 3p(n)$, since it is equivalent to $\underbrace{999\ldots9}_{d\text{times}} = 3 \cdot 9^d$. □ □

It is also interesting to study other special cases of the equation

$$n = kp(n).$$

In this connection, we rise the problem to determine the numbers n such that n is divisible by both $p(n)$ and $s(n)$, or, even more restrictively, $n \,\vdots\, p(n)s(n)$. There exist infinitely many numbers n with this property. For instance, for $n = 11\ldots1$ with $d(n) = 3^i$, we have $n = (10^d - 1)/9$, $p(n) = 1$, $s(n) = 3^i$ and $n \,\vdots\, p(n)s(n)$, since $10^{3^i} - 1 \,\vdots\, 3^{i+2}$. (The latter fact can be proved by induction.)

However, if we impose the condition

$$n = p(n)s(n), \tag{5}$$

then we find for $n < 10^7$ only the solutions 1, 135 and 144. In view of inequality (3) and of $s(n) \le 9d$, it is easy to show that, if n satisfies (5), then $d(n) \le 60$. Thus (5) holds only for finitely many values of n.

Open problem. Determine all the natural numbers n having property (5).

References

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