NNTDM 10 (2004), 3, 57-67

On meet and join matrices on A-sets and related sets

Ismo Korkee

Department of Mathematics, Statistics and Philosophy
University of Tampere
FI-33014 University of Tampere, Finland
e-mail: csisko@uta.fi

Abstract

Let $(P, \wedge)$ be a meet-semilattice and let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P. We say that S is an A-set if $A = \{x_i \wedge x_j \mid x_i \neq x_j\}$ is a chain. For example, chains and a-sets (with $A = \{a\}$) are known trivial A-sets. The meet matrix $(S)_f$ on S with respect to a function $f : P \to \mathbf{C}$ is defined as $((S)_f)_{ij} = f(x_i \wedge x_j)$.

We present a recursive structure theorem for meet matrices on A-sets and thus obtain a recursive formula for $\det(S)_f$ and for $(S)_f^{-1}$ on A-sets. The recursive formulae also yield explicit formulae, e.g. the known determinant and inverse formulae on chains and a-sets. We also present the dual forms of our results, i.e. the determinant formulae and the inverse formulae for join matrices on join-semilattices. Finally, we suggest how our results can be generalized to more complicated cases.

As special cases these results hold also for GCD and LCM matrices and for their unitary analogies GCUD and LCUM matrices.

Key words and phrases: Meet matrix, Join matrix, Determinant, Inverse matrix, A-sets, GCD matrix, LCM matrix.

AMS subject classifications: 11C20, 15A09, 15A15.

1 Introduction

Let $(P, \leq) = (P, \wedge, \vee)$ be a locally finite lattice, let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P and let $f : P \to \mathbf{C}$ be a function. The meet matrix $(S)_f$ and the join matrix $[S]_f$ on S with respect to f are defined as $((S)_f)_{ij} = f(x_i \wedge x_j)$ and $([S]_f)_{ij} = f(x_i \vee x_j)$.

Haukkanen [4] introduced meet matrices $(S)_f$ and obtained formulae for $\det(S)_f$ and $(S)_f^{-1}$ (see also [12] and [13]). Korkee and Haukkanen [9] used incidence functions (see Section 2) in the study of meet matrices. There we obtained new upper and lower bounds for $\det(S)_f$ and a new formula for $(S)_f^{-1}$ on meet-closed sets S (i.e., $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$). In [10] we introduced join matrices and presented formulae for $\det[S]_f$, new upper and lower bounds for $\det[S]_f$ and a new formula for $[S]_f^{-1}$ on join-closed sets S (i.e., $x_i, x_j \in S \Rightarrow$

$x_i \vee x_j \in S$). By assuming the semi-multiplicativity of f, formulae for $\det(S)_f$ and $(S)_f^{-1}$ on join-closed sets and formulae for $\det[S]_f$ and $[S]_f^{-1}$ on meet-closed sets are also presented in [10]. Recently Korkee and Haukkanen [11] presented a new method for calculating $\det(S)_f$ and $(S)_f^{-1}$ on those sets S which are not necessarily meet-closed.

We say that S is an A-set if the set $A = \{x_i \wedge x_j \mid x_i \neq x_j\}$ is a chain (an A-set need not be meet-closed). For example, chains and a-sets (with $A = \{a\}$) are known trivial A-sets. Note that in the literature these sets have not previously been categorized in the same subclass. Since the method, presented in [11], adapted to A-sets might not be sufficiently effective, we give a new structure theorem for $(S)_f$ where S is an A-set. One of its features is that it supports recursive function calls.

By the structure theorem we obtain a recursive formula for $\det(S)_f$ and for $(S)_f^{-1}$ on A-sets. By dissolving the recursion on certain sets we also obtain e.g. the known explicit determinant and inverse formulae on chains and a-sets. We also briefly list the dual forms of our results, i.e. the structure theorem, determinant formulae and the inverse formulae for join matrices on join-semilattices. Finally, we suggest how our results can be generalized to more complicated cases. The elements $x_1, x_2, \ldots, x_n$ of S can be replaced with a family $S_1, S_2, \ldots, S_n$ of subsets of P. This replacement works if $x_i = \min S_i$ and $\{x_i \wedge x_j \mid x_i \in S_i,\ x_j \in S_j\}$ fulfills some conditions noted later.

Note that $(\mathbf{Z}_+, |) = (\mathbf{Z}_+, \gcd, \mathrm{lcm})$ is a locally finite lattice, where $|$ is the usual divisibility relation and gcd and lcm stand for the greatest common divisor and the least common multiple of integers. Thus meet and join matrices are generalizations of GCD matrices $((S)_f)_{ij} = f(\gcd(x_i, x_j))$ and LCM matrices $([S]_f)_{ij} = f(\mathrm{lcm}(x_i, x_j))$ and therefore the results in this paper also hold for GCD and LCM matrices. For general accounts of GCD and LCM matrices, see [6] and [10, Section 6]. Meet and join matrices are also generalizations of GCUD and LCUM matrices, the unitary analogies of GCD and LCM matrices, see [5] and [8]. Thus the results also hold for GCUD and LCUM matrices (provided that we define LCUM matrices as done in [8]).

2 Definitions

Let $(P, \leq) = (P, \wedge)$ be a meet-semilattice and let S be a nonempty subset of P. We say that S is meet-closed if $x \wedge y \in S$ whenever $x, y \in S$. We say that S is lower-closed if $(x \in S, y \leq x) \Rightarrow y \in S$ holds for every $y \in P$. It is clear that a lower-closed set is always meet-closed but the converse is not true. We review the dual concepts on join-semilattices in Section 4.

The method used requires that we arrange the elements of S analogously to the elements of chain A. Thus we give a more applicable definition for a-sets and A-sets than were seen in Introduction.

Definition 2.1 *Define the binary operation* $\sqcap$ *by*

$$S_1 \sqcap S_2 = \{x \wedge y \mid x \in S_1,\ y \in S_2,\ x \neq y\}, \tag{2.1}$$

where S_1 *and* S_2 *are nonempty subsets of* P*. Let* S *be a subset of* P *and let* $a \in P$*. If* $S \sqcap S = \{a\}$*, then the set* S *is said to be an* a*-set.*

Definition 2.2 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P with $x_i < x_j \Rightarrow i < j$ and let $A = \{a_1, a_2, \ldots, a_{n-1}\}$ be a multichain (i.e. a chain where duplicates are allowed) with $a_1 \leq a_2 \leq \cdots \leq a_{n-1}$. The set S is said to be an A-set if $\{x_k\} \sqcap \{x_{k+1}, \ldots, x_n\} = \{a_k\}$ for all $k = 1, 2, \ldots, n-1$.*

Note that every chain $S = \{x_1, x_2, \ldots, x_n\}$ is an A-set with $A = S \setminus \{x_n\}$ and every a-set is always an A-set with $A = \{a\}$.

As far as we know, the concept of an A-set has not previously been presented in the literature. The concept of an a-set is introduced e.g. in [4].

Definition 2.3 *Let f be a complex-valued function on P. Then the $n \times n$ matrix $(S)_f$, where $((S)_f)_{ij} = f(x_i \wedge x_j)$, is called the meet matrix on S with respect to f.*

In what follows, let $S = \{x_1, x_2, \ldots, x_n\}$ always be a finite subset of P with $x_i < x_j \Rightarrow i < j$. Let also $A = \{a_1, a_2, \ldots, a_m\}$ with $a_i < a_j \Rightarrow i < j$. Note that S has always n distinct elements, but it is possible that the set A is a multiset. Let f be a complex-valued function on P.

3 Meet matrices on A-sets

3.1 Structure Theorem

Recently Korkee and Haukkanen [11] presented a new method for calculating $\det(S)_f$ and $(S)_f^{-1}$ on those sets S which are not meet-closed (an A-set need not be meet-closed). The method may not be sufficiently effective on A-sets. Therefore we present a new structure theorem for meet matrices on A-sets which supports recursive function calls.

Theorem 3.1 (Structure Theorem) *Let $S = \{x_1, x_2, \ldots, x_n\}$ be an A-set, where $A = \{a_1, a_2, \ldots, a_{n-1}\}$ is a multichain. Let $f_1, f_2, \ldots, f_n$ denote the functions on P defined by $f_1 = f$ and*

$$f_{k+1}(x) = f_k(x) - \frac{f_k(a_k)^2}{f_k(x_k)} \tag{3.1}$$

for $k = 1, 2, \ldots, n-1$. Then

$$(S)_f = M^{\mathrm{T}} D M, \tag{3.2}$$

where $D = \mathrm{diag}(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$ and M is the $n \times n$ upper triangular matrix with 1's on its main diagonal, and further

$$(M)_{ij} = \frac{f_i(a_i)}{f_i(x_i)} \tag{3.3}$$

for all $i < j$. (Note that $f_1, \ldots, f_n$ exist if and only if $(f_k(x_k) = 0, a_k \neq x_k) \Rightarrow f_k(a_k) = 0$ holds for all $k = 1, 2, \ldots, n-1$. In the case $f_k(a_k) = f_k(x_k) = 0$ we can write e.g. $(M)_{kj} = 0$ for all $k < j$.)

Proof Let $i < j$. Then

$$(M^{\mathrm{T}}DM)_{ij} = \sum_{k=1}^{n}(M)_{ki}(D)_{kk}(M)_{kj} = f_i(a_i) + \sum_{k=1}^{i-1}\frac{f_k(a_k)^2}{f_k(x_k)} \tag{3.4}$$

$$= f_i(a_i) + \sum_{k=1}^{i-1}(f_k(a_i) - f_{k+1}(a_i)) = f_1(a_i) = f(x_i \wedge x_j).$$

The case $i = j$ is similar, we only replace every a_i with x_i in (3.4). Since $M^{\mathrm{T}}DM$ is symmetric, we do not need to treat the case $i > j$. □

3.2 Determinant of meet matrix on A-sets

By Structure Theorem we obtain a new recursive formula for $\det(S)_f$ on A-sets.

Theorem 3.2 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be an A-set, where $A = \{a_1, a_2, \ldots, a_{n-1}\}$ is a multichain. Let $f_1, f_2, \ldots, f_n$ be the functions defined in* (3.1). *Then*

$$\det(S)_f = f_1(x_1)f_2(x_2)\cdots f_n(x_n). \tag{3.5}$$

By Theorem 3.2 we obtain a known explicit formula for $\det(S)_f$ on chains presented in [4, Corollary 3] and [13, Corollary 1].

Corollary 3.1 *If $S = \{x_1, x_2, \ldots, x_n\}$ is a chain, then*

$$\det(S)_f = f(x_1)\prod_{k=2}^{n}(f(x_k) - f(x_{k-1})). \tag{3.6}$$

Proof By Theorem 3.2 we have $\det(S)_f = f_1(x_1)f_2(x_2)\cdots f_n(x_n)$, where $f_1 = f$ and $f_{k+1}(x) = f_k(x) - f_k(x_k) = f(x) - f(x_k)$ for all $k = 1, 2, \ldots, n-1$. This completes the proof. □

By Theorem 3.2 we also obtain a known explicit formula for $\det(S)_f$ on a-sets. This formula has been presented (with different notation) in [4, Corollary of Theorem 3] and [11, Corollaries 5.1 and 5.2], and also in [2, Theorem 3] in number-theoretic setting.

Note that the case $f(a) = 0$ is trivial, since then $(S)_f = \mathrm{diag}(f(x_1), f(x_2), \ldots, f(x_n))$ and $\det(S)_f = f(x_1)f(x_2)\cdots f(x_n)$.

Corollary 3.2 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be an a-set, where $f(a) \neq 0$. If $a \in S$ (i.e. $a = x_1$), then*

$$\det(S)_f = f(a)(f(x_2) - f(a))\cdots(f(x_n) - f(a)). \tag{3.7}$$

If $a \notin S$, then

$$\begin{aligned}\det(S)_f &= \sum_{k=1}^{n}\frac{f(a)(f(x_1) - f(a))\cdots(f(x_n) - f(a))}{f(x_k) - f(a)} \\ &\quad + (f(x_1) - f(a))\cdots(f(x_n) - f(a)).\end{aligned} \tag{3.8}$$

Proof Let $a = x_1$. After reducing $(S)_f$ by appropriate row operations, we note that (3.7) holds. We prove the case $a < x_1$ by induction on n. The claim holds clearly for $n = 2$. State the inductive hypothesis that the claim holds for $n = m$ and consider the a-set $S_{m+1} = \{x_1, \dots, x_m, x_{m+1}\}$. Rename the elements of S_{m+1} such that $f(x_1) \neq 0$ if it is necessary. Since S_{m+1} is an a-set, by Theorem 3.2 we have $\det(S_{m+1})_f = f_1(x_1)f_2(x_2)\cdots f_{m+1}(x_{m+1}) = f(x_1)\det(S_m)_g$, where $S_m = \{x_2, \dots, x_{m+1}\}$ and $g(x) = f(x) - f(a)^2/f(x_1)$ for all $x \in P$. (Note that the existence of $f_2, f_3 \dots, f_{m+1}$ is irrelevant, since $\det(S_m)_g$ exists anyway.) Now, adapting the inductive hypothesis on $\det(S_m)_g$ we have

$$\begin{aligned}\det(S_{m+1})_f &= \sum_{k=2}^{m+1} \frac{f(x_1)g(a)(g(x_2)-g(a))\cdots(g(x_{m+1})-g(a))}{g(x_k)-g(a)} \\ &\quad + (f(x_1)-f(a))(g(x_2)-g(a))\cdots(g(x_{m+1})-g(a)) \\ &\quad + f(a)(g(x_2)-g(a))\cdots(g(x_{m+1})-g(a)). \qquad (3.9)\end{aligned}$$

Since $g(x_k) - g(a) = f(x_k) - f(a)$ for all $k = 2, 3, \dots, m+1$ and further $f(x_1)g(a) = f(a)(f(x_1) - f(a))$, by (3.9) we see that the claim holds for $n = m+1$. This completes the proof by induction. □

In [2], [4] and [11] the formula (3.8) has been written in the form

$$\begin{aligned}\det(S)_f &= f(a)(f(x_1)-f(a))\cdots(f(x_n)-f(a)) \qquad (3.10)\\ &\quad \times \Big(\frac{1}{f(a)} + \frac{1}{f(x_1)-f(a)} + \cdots + \frac{1}{f(x_n)-f(a)}\Big).\end{aligned}$$

Obviously we obtain (3.8) if we expand (3.10). Although the question of the denominators being zeros is irrelevant, we prefer to use the form (3.8).

The formula for $\det(S)_f$ on a-sets can also be stated as in the next corollary. The proof is left to the reader as an application of induction and multinomial formulae, see [1]. The Cauchy-Binet formula and the definition of the determinant [7] also yield the same result. Note that there is no need to calculate B_{n-1} in (3.12).

Corollary 3.3 *Let $S = \{x_1, x_2, \dots, x_n\}$ be an a-set, where $f(a) \neq 0$. If $a \in S$ (i.e. $a = x_1$), then*

$$\det(S)_f = \sum_{k=0}^{n} (-1)^{n-k-1} f(a)^{n-k} B_k, \qquad (3.11)$$

where $B_0 = 1$ and $B_k = \sum_{2 \le i_1 < \cdots < i_k \le n} f(x_{i_1})f(x_{i_2})\cdots f(x_{i_k})$ for all $k = 1, 2, \dots, n$. If $a \notin S$, then

$$\det(S)_f = \sum_{k=0}^{n} (-1)^{n-k-1}(n-k-1) f(a)^{n-k} B_k, \qquad (3.12)$$

where $B_0 = 1$ and $B_k = \sum_{1 \le i_1 < \cdots < i_k \le n} f(x_{i_1})f(x_{i_2})\cdots f(x_{i_k})$ for all $k = 1, 2, \dots, n$.

By Theorem 3.2 it may also be possible to obtain explicit formulae for $\det(S)_f$ on other specialized A-sets. However, we next give an example of how to apply our recursive formula.

Example 3.1 Let $(P, \leq) = (\mathbf{Z}_+, |)$ and $S = \{7, 10, 12, 18\}$. Since S is an A-set with the chain $A = \{1, 2, 6\}$, by (3.1) we have $f_1 = f$, $f_2(x) = f_1(x) - f_1(1)^2/f_1(7)$, $f_3(x) = f_2(x) - f_2(2)^2/f_2(10)$ and $f_4(x) = f_3(x) - f_3(6)^2/f_3(12)$. Let $f(x) = x$. Then $f_1(x) = x$, $f_2(x) = x - \frac{1}{7}$, $f_3(x) = x - \frac{34}{69}$, $f_4(x) = x - \frac{1242}{397}$ and by Theorem 3.2 we have

$$\det(S)_f = f_1(7)f_2(10)f_3(12)f_4(18) = 7(\tfrac{69}{7})(\tfrac{794}{69})(\tfrac{5904}{397}) = 11808.$$

A corresponding recursive procedure is easy to program with mathematical software that is capable of symbolic calculation. For example, by programming with Mathematica™ system as

```
S = {7, 10, 12, 18}; A = {1, 2, 6};
f[1, x_] := x;
f[k_, x_] := f[k - 1, x] - f[k - 1, A[[k - 1]]]^2/f[k - 1, S[[k - 1]]];
detSf = Product[f[k, S[[k]]], {k, Length[S]}]
```

we obtain the answer to our problem and an idea to compute $\det(S)_f$ on any A-set and any function f. Note that the given procedure is only a draft and dividing by zero, for example, is not concerned.

3.3 Inverse of meet matrix on A-sets

By Structure Theorem we obtain a new recursive formula for $(S)_f^{-1}$ on A-sets.

Theorem 3.3 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be an A-set, where $A = \{a_1, a_2, \ldots, a_{n-1}\}$ is a multichain. Let $f_1, f_2, \ldots, f_n$ be the functions defined in* (3.1)*, where $f_i(x_i) \neq 0$ for $i = 1, 2, \ldots, n$. Then $(S)_f$ is invertible and*

$$(S)_f^{-1} = N\Delta N^{\mathrm{T}}, \tag{3.13}$$

where $\Delta = \mathrm{diag}(1/f_1(x_1), 1/f_2(x_2), \ldots, 1/f_n(x_n))$ and N is the $n \times n$ upper triangular matrix with 1's on its main diagonal, and further

$$(N)_{ij} = -\frac{f_i(a_i)}{f_i(x_i)} \prod_{k=i+1}^{j-1} \Big(1 - \frac{f_k(a_k)}{f_k(x_k)}\Big) \tag{3.14}$$

for all $i < j$.

Proof By Structure Theorem we have $(S)_f = M^{\mathrm{T}}DM$, where M is the matrix defined in (3.3) and $D = \mathrm{diag}(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$. Therefore $(S)_f^{-1} = N\Delta N^{\mathrm{T}}$, where $D^{-1} = \mathrm{diag}(1/f_1(x_1), 1/f_2(x_2), \ldots, 1/f_n(x_n))$ and $M^{-1} = N$ is the $n \times n$ upper triangular matrix in (3.14). □

By Theorem 3.3 we obtain a formula for $(S)_f^{-1}$ on chains. This formula has not previously been presented in the literature but is, however, easy to find by reducing $(S)_f$ through suitable row operations.

Corollary 3.4 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a chain. Let $f(x_1) \neq 0$ and $f(x_{k+1}) \neq f(x_k)$ for all $k = 1, 2, \ldots, n-1$. Then $(S)_f$ is invertible and*

$$((S)_f^{-1})_{ij} = \begin{cases} \frac{1}{f(x_1)} + \frac{1}{f(x_2)-f(x_1)} & \text{if } i = j = 1, \\ \frac{1}{f(x_{i+1})-f(x_i)} + \frac{1}{f(x_i)-f(x_{i-1})} & \text{if } 1 < i = j < n, \\ \frac{1}{f(x_n)-f(x_{n-1})} & \text{if } i = j = n, \\ \frac{1}{f(x_k)-f(x_{k+1})} & \text{if } k = i = j-1 \text{ or } k = j = i-1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

Proof Now $f_1 = f$ and $f_{k+1}(x) = f_k(x) - f_k(x_k) = f(x) - f(x_k)$ for all $k = 1, 2, \ldots, n-1$. Thus by Theorem 3.3 we have $(S)_f^{-1} = N\Delta N^{\mathrm{T}}$, where Δ is the diagonal matrix with $(\Delta)_{11} = 1/f(x_1)$ and

$$(\Delta)_{kk} = [f_k(x_k)]^{-1} = [f_{k-1}(x_k) - f_{k-1}(x_{k-1})]^{-1} = [f(x_k) - f(x_{k-1})]^{-1} \tag{3.16}$$

for $k = 2, 3, \ldots, n$. Further, by (3.14) N is the $n \times n$ upper triangular matrix with 1's on its main diagonal and

$$(N)_{ij} = \begin{cases} -1 & \text{if } i = j-1, \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

for $i < j$. Thus by calculating $N\Delta N^{\mathrm{T}}$ we obtain (3.15). □

By Theorem 3.3 we obtain a known formula for $(S)_f^{-1}$ on a-sets presented in [11, Corollaries 5.1 and 5.2].

Corollary 3.5 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be an a-set, where $f(a) \neq 0$ and $f(x_k) \neq f(a)$ for all $k = 2, \ldots, n$. If $a \in S$ (i.e. $a = x_1$), then $(S)_f$ is invertible and*

$$((S)_f^{-1})_{ij} = \begin{cases} \frac{1}{f(a)} + \sum_{k=2}^{n} \frac{1}{f(x_k)-f(a)} & \text{if } i = j = 1, \\ \frac{1}{f(x_i)-f(a)} & \text{if } 1 < i = j, \\ \frac{1}{f(a)-f(x_k)} & \text{if } 1 = i < j = k \text{ or } 1 = j < i = k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.18}$$

If $a \notin S$ and further $f(x_1) \neq f(a)$ and $\frac{1}{f(a)} \neq \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}$, then $(S)_f$ is invertible and

$$((S)_f^{-1})_{ij} = \begin{cases} \frac{1}{f(x_i)-f(a)} - \frac{1}{[f(x_i)-f(a)]^2}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & \text{if } i = j, \\ -\frac{1}{[f(x_i)-f(a)]\,[f(x_j)-f(a)]}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & \text{if } i \neq j. \end{cases} \tag{3.19}$$

Proof Let $a = x_1$. By (3.1) we have $f_1 = f$, $f_2(x) = f(x) - f(a)$ and $f_2 = f_3 = \cdots = f_n$. Since $f_i(x_i) \neq 0$ and $f_i(a_i) = 0$ for all $i = 1, \ldots, n$, by Theorem 3.3 we have $(S)_f^{-1} = N\Delta N^{\mathrm{T}}$, where

$$\Delta = \mathrm{diag}\Big(\frac{1}{f(a)}, \frac{1}{f(x_2) - f(a)}, \ldots, \frac{1}{f(x_n) - f(a)}\Big) \tag{3.20}$$

and N is the upper triangular matrix with 1's on its main diagonal and

$$(N)_{ij} = -\frac{f_i(a_i)}{f_i(x_i)} \prod_{k=i+1}^{j-1} \Big(1 - \frac{f_k(a_k)}{f_k(x_k)}\Big) = \begin{cases} -1 & \text{if } i = 1 \text{ and } j \geq 2, \\ 0 & \text{otherwise} \end{cases} \tag{3.21}$$

for $i < j$. Thus by calculating $N\Delta N^{\mathrm{T}}$ we obtain (3.18). Let $a \neq x_1$. By adapting (3.18) to the a-set $S' = \{a, x_1, x_2, \ldots, x_n\}$ we have

$$(S')_f = \begin{bmatrix} f(a) & \mathbf{e} \\ \mathbf{e}^{\mathrm{T}} & (S)_f \end{bmatrix}, \qquad (S')_f^{-1} = \begin{bmatrix} s & \mathbf{g} \\ \mathbf{g}^{\mathrm{T}} & H \end{bmatrix},$$

where $\mathbf{e} = (f(a), \ldots, f(a))$, $s = 1/f(a) + \sum_{i=1}^{n}(f(x_i) - f(a))^{-1}$, $\mathbf{g} = ((f(a) - f(x_1))^{-1}, \ldots, (f(a) - f(x_n))^{-1})$ and $H = \mathrm{diag}((f(x_1) - f(a))^{-1}, \ldots, (f(x_n) - f(a))^{-1})$. By method used in [11, Theorem 3.3] or [14, Section 2.2] we know that $(S)_f = H - \mathbf{g}^{\mathrm{T}}\mathbf{g}/s$, which gives (3.19). □

Korkee and Haukkanen [9, Theorem 7.1] presented the inverse formula

$$((S)_f^{-1})_{ij} = \sum_{x_i \le x_k;\ x_j \le x_k} \frac{\mu_S(x_i, x_k)\mu_S(x_j, x_k)}{\sum\limits_{z \le x_k;\ z \not\le x_1, \ldots, x_{k-1}} (f * \mu)(z)} \tag{3.22}$$

for $(S)_f^{-1}$ on meet-closed sets. Here $*$ is the convolution of incidence functions of P, ζ is the zeta function of P, μ is the inverse of ζ and $\mu_S = (\zeta_S)^{-1}$, where ζ_S is the restriction of ζ on $S \times S$, see [1, p. 138–151]. Since every chain and a-set with $a \in S$ is meet-closed, (3.22) is expected to yield the same result as Corollaries 3.4 and 3.5. This is certainly true, since if we calculate N^{-1} from (3.17) (resp. from (3.21)), we obtain the upper triangular matrix $M = N^{-1}$ with $(M)_{ij} = 1$ whenever $i \le j$ (resp. $M = N^{-1}$ with $(M)_{ij} = 1$ whenever $i = j$ or $1 = i < j$). In both cases $M = [\zeta_S(x_i, x_j)]$ and thus $N = [\mu_S(x_i, x_j)]$. The denominator in (3.22) and Δ in (3.16) (resp. Δ in (3.20)) are also easy to prove equal.

4 Join matrices on duals of A-sets

From the perspective of lattice theory the results of meet matrices on meet-semilattices obviously hold for join matrices on join-semilattices. In this section we briefly present these dual results of the results found in earlier sections. We omit the proofs, since they are based on the renumbering of the elements of S backwards, see e.g. [10].

In this section let $(P, \le) = (P, \vee)$ be a join-semilattice (with $x \le y \Leftrightarrow x \vee y = y$, see [3]). We say that S is join-closed if $x \vee y \in S$ whenever $x, y \in S$. We say that S is upper-closed if $(x \in S, x \le y) \Rightarrow y \in S$ holds for every $y \in P$. It is clear that an upper-closed set is always join-closed but the converse does not hold. Since we want to keep the terminology clear, we only give the following definitions.

Definition 4.1 *Define the binary operation* $\sqcup$ *by*

$$S_1 \sqcup S_2 = \{x_1 \vee x_2 \mid x_1 \in S_1,\ x_2 \in S_2,\ x_1 \neq x_2\} \tag{4.1}$$

whenever S_1 *and* S_2 *are nonempty subsets of* P.

Definition 4.2 *Let* f *be a complex-valued function on* P. *Then the* $n \times n$ *matrix* $[S]_f$, *where* $([S]_f)_{ij} = f(x_i \vee x_j)$, *is called the join matrix on* S *with respect to* f.

Theorem 4.1 (Structure Theorem) *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite set such that $\{x_1, x_2, \ldots, x_{k-1}\} \sqcup \{x_k\} = \{a_k\}$ for all $k = 2, 3, \ldots, n$, where $A = \{a_2, a_3, \ldots, a_n\}$ is a multichain. Let $f_1, f_2, \ldots, f_n$ denote the functions on P defined by $f_1 = f$ and*

$$f_{k+1}(x) = f_k(x) - f_k(a_{n-k+1})^2 / f_k(x_{n-k+1}) \tag{4.2}$$

whenever $k = 1, 2, \ldots, n-1$. Then

$$[S]_f = M^{\mathrm{T}} D M, \tag{4.3}$$

where $D = \mathrm{diag}(f_n(x_1), f_{n-1}(x_2), \ldots, f_1(x_n))$ and M is the $n \times n$ lower triangular matrix with 1's on its main diagonal, and further

$$(M)_{ij} = f_{n-i+1}(a_i)/f_{n-i+1}(x_i) \tag{4.4}$$

for all $i > j$.

Theorem 4.2 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set such that $\{x_1, x_2, \ldots, x_{k-1}\} \sqcup \{x_k\} = \{a_k\}$ for all $k = 2, 3, \ldots, n$, where $A = \{a_2, a_3, \ldots, a_n\}$ is a multichain. Let $f_1, f_2, \ldots, f_n$ be the functions defined in (4.2). Then*

$$\det[S]_f = f_n(x_1) f_{n-1}(x_2) \cdots f_1(x_n). \tag{4.5}$$

Corollary 4.1 [13, Corollary 1] *If $S = \{x_1, x_2, \ldots, x_n\}$ is a chain, then*

$$\det[S]_f = f(x_n) \prod_{k=2}^{n} (f(x_{k-1}) - f(x_k)). \tag{4.6}$$

Corollary 4.2 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set such that $x_i \vee x_j = a$ whenever $x_i \neq x_j$ and let $f(a) \neq 0$. If $a \in S$ (i.e. $a = x_n$), then*

$$\det[S]_f = (f(x_1) - f(a)) \cdots (f(x_{n-1}) - f(a)) f(a). \tag{4.7}$$

If $a \notin S$, then

$$\begin{aligned} \det[S]_f \quad = \quad & \sum_{k=1}^{n} \frac{(f(x_1) - f(a)) \cdots (f(x_n) - f(a)) f(a)}{f(x_k) - f(a)} \\ & + (f(x_1) - f(a)) \cdots (f(x_n) - f(a)). \end{aligned} \tag{4.8}$$

Theorem 4.3 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set such that $\{x_1, x_2, \ldots, x_{k-1}\} \sqcup \{x_k\} = \{a_k\}$ for all $k = 2, 3, \ldots, n$, where $A = \{a_2, a_3, \ldots, a_n\}$ is a multichain. Let $f_1, f_2, \ldots, f_n$ be the functions defined in (4.2), where $f_{n-i+1}(x_i) \neq 0$ for $i = 1, 2, \ldots, n$. Then $[S]_f$ is invertible and*

$$[S]_f^{-1} = N \Delta N^{\mathrm{T}}, \tag{4.9}$$

where $\Delta = \mathrm{diag}(1/f_n(x_1), 1/f_{n-1}(x_2), \ldots, 1/f_1(x_n))$ and N is the $n \times n$ lower triangular matrix with 1's on its main diagonal and

$$(N)_{ij} = -\frac{f_{n-i+1}(a_i)}{f_{n-i+1}(x_i)} \prod_{k=j-1}^{i+1} \Big(1 - \frac{f_{n-k+1}(a_k)}{f_{n-k+1}(x_k)}\Big) \tag{4.10}$$

for all $i > j$.

Corollary 4.3 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a chain. If $f(x_n) \neq 0$ and $f(x_{k-1}) \neq f(x_k)$ for all $k = 2, 3, \ldots, n$, then $[S]_f$ is invertible and*

$$([S]_f^{-1})_{ij} = \begin{cases} \frac{1}{f(x_1)-f(x_2)} & \text{if } i = j = 1, \\ \frac{1}{f(x_i)-f(x_{i+1})} + \frac{1}{f(x_{i-1})-f(x_i)} & \text{if } 1 < i = j < n, \\ \frac{1}{f(x_n)} + \frac{1}{f(x_{n-1})-f(x_n)} & \text{if } i = j = n, \\ \frac{1}{f(x_{k+1})-f(x_k)} & \text{if } k = i = j-1 \text{ or } k = j = i-1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Corollary 4.4 *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set such that $x_i \vee x_j = a$ whenever $x_i \neq x_j$ and let $f(a) \neq 0$ and $f(x_k) \neq f(a)$ for all $k = 2, \ldots, n-1$. If $a \in S$ (i.e. $a = x_n$), then $[S]_f$ is invertible and*

$$([S]_f^{-1})_{ij} = \begin{cases} \frac{1}{f(x_i)-f(a)} & \text{if } i = j < n, \\ \frac{1}{f(a)} + \sum_{k=1}^{n-1} \frac{1}{f(x_k)-f(a)} & \text{if } i = j = n, \\ \frac{1}{f(a)-f(x_k)} & \text{if } k = i < j = n \text{ or } k = j < i = n, \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

If $a \notin S$ and further $\frac{1}{f(a)} \neq \sum_{k=1}^{n-1} \frac{1}{f(x_k)-f(a)}$, then $[S]_f$ is invertible and

$$([S]_f^{-1})_{ij} = \begin{cases} \frac{1}{f(x_i)-f(a)} - \frac{1}{[f(x_i)-f(a)]^2}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & \text{if } i = j, \\ -\frac{1}{[f(x_i)-f(a)]\,[f(x_j)-f(a)]}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & \text{if } i \neq j. \end{cases} \tag{4.13}$$

5 Further generalizations

In this section let $(P, \leq) = (P, \wedge)$ again be a meet-semilattice and let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of P. The original idea of finding $\det(S)_f$ on A-sets is based on reducing $(S)_f$ by appropriate row operations (which further leads to Structure Theorem). In the next theorem we show that the idea can be further generalized by replacing the elements $x_1, x_2, \ldots, x_n$ of S with a family $S_1, S_2, \ldots, S_n$ of subsets of P.

Theorem 5.1 *Let $S = \cup_{k=1}^{n} S_i$ be a union of disjoint subsets of P such that there exists* $\min S_k = x_k$*, and $S_k \sqcap S_{k+1} = \{a_k\}$ for all $k = 1, 2, \ldots, n-1$, where $A = \{a_1, a_2, \ldots, a_{n-1}\}$ is a multichain. If the functions $f_1, f_2, \ldots, f_n$ on P defined in* (3.1) *exists, then*

$$\det(S)_f = \det(S_1)_{f_1} \det(S_2)_{f_2} \cdots \det(S_n)_{f_n}. \tag{5.1}$$

(Note that $\{x_1, x_2, \ldots, x_n\}$ is an A-set.)

Proof We prove the claim by induction on n. Let $n = 2$. After reducing $(S)_f$ by appropriate row operations we have

$$\det(S)_f = \begin{vmatrix} (S_1)_f & B^{\mathrm{T}} \\ B & (S_2)_f \end{vmatrix} = \begin{vmatrix} (S_1)_f & B^{\mathrm{T}} \\ O & (S_2)_f - C \end{vmatrix}, \tag{5.2}$$

where every element of the $k \times m$ matrix B is $f(a_1)$, O being the $k \times m$ zero matrix, and every element of the $k \times k$ matrix C equals $f(a_1)^2/f(x_1)$. Note that since $x_1 = \min S_1$, the first row of $(S_1)_f$ is $(f(x_1), f(x_1), \ldots, f(x_1))^{\mathrm{T}}$. Thus by (5.2) we have $\det(S)_f = \det(S_1)_f \det((S_2)_f - C)$. Since $f(x_1) \neq 0$, we can define $f_1 = f$ and $f_2(x) = f_1(x) - f_1(a_1)^2/f_1(x_1)$ for all $x \in P$. Thus $\det(S)_f = \det(S_1)_{f_1} \det(S_2)_{f_2}$ and the claim is true for $n = 2$. We state the inductive hypothesis that Theorem 5.1 holds for $n = m$. Consider the case $n = m + 1$ and let $T_{m+1} = S_1 \cup S_2 \cup \cdots \cup S_{m+1}$ with the multichain $A = \{a_1, a_2, \ldots, a_m\}$ such that the assumptions of Theorem 5.1 are fulfilled. Since $S_1 \sqcap T_m = \{a_1\}$, where $T_m = S_2 \cup \cdots \cup S_{m+1}$, by the first step of induction we have $\det(T_{m+1})_f = \det(S_1)_{f_1} \det(T_m)_{f_2}$, where $f_1 = f$ and $f_2(x) = f_1(x) - f_1(a_1)^2/f_1(x_1)$. Further, since the inductive hypothesis holds for T_m with the multichain $A \setminus \{a_1\}$, then $\det(T_m)_{f_2} = \det(S_2)_{f_2} \det(S_3)_{f_3} \cdots \det(S_{m+1})_{f_{m+1}}$, where $f_{k+1}(x) = f_k(x) - f_k(a_k)^2/f_k(x_k)$ for all $k = 1, \ldots, m$. Thus the claim holds for $n = m + 1$. This completes the proof by induction. □

Acknowledgement The author wishes to thank Pentti Haukkanen for his reading of the manuscript and kind comments.

References

[1] M. Aigner, *Combinatorial Theory.* Springer–Verlag, 1979.

[2] S. Beslin and S. Ligh, GCD-closed sets and the determinants of GCD matrices, *Fibonacci Quart.*, 30: 157–160 (1992).

[3] G. Birkhoff, *Lattice Theory.* American Mathematical Society Colloquium Publications, 25, Rhode Island, (1984).

[4] P. Haukkanen, On meet matrices on posets, *Linear Algebra Appl.* 249: 111–123 (1996).

[5] P. Haukkanen and J. Sillanpää, Some analogues of Smith's determinant, *Linear and Multilinear Algebra* 41: 233–244 (1996).

[6] P. Haukkanen, J. Wang and J. Sillanpää, On Smith's determinant, *Linear Algebra Appl.* 258: 251-269 (1997).

[7] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[8] I. Korkee, A note on meet and join matrices and their special cases GCUD and LCUM matrices. Accepted to *Int. J. Pure Appl. Math.*

[9] I. Korkee and P. Haukkanen, Bounds for determinants of meet matrices associated with incidence functions, *Linear Algebra Appl.* 329(1–3): 77–88 (2001).

[10] I. Korkee and P. Haukkanen, On meet and join matrices associated with incidence functions. *Linear Algebra Appl.*, 372: 127–153 (2003).

[11] I. Korkee and P. Haukkanen, On meet matrices with respect to reduced, extended and exchanged sets. Accepted to *JP J. Algebra Number Theory Appl.*

[12] B. V. Rajarama Bhat, On greatest common divisor matrices and their applications, *Linear Algebra Appl.* 158: 77–97 (1991).

[13] B. Wang, Explicit Expressions of Smith's Determinant on a Poset, *Acta Math. Sin. (Engl. Ser.)*, 17(1): 161–168 (2001).

[14] F. Zhang, Matrix theory. Basic results and techniques, Universitext, Springer-Verlag, New York, 1999.