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SOME PROPERTIES OF FERMATIAN NUMBERS

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ABSTRACT

This paper looks at some basic number theoretic properties of Fermatian numbers. We define the n-th reduced Fermatian number in terms of

$$\underline{q}_n = \begin{cases} -q^n \underline{q}_{-n} & (n < 0) \\ 1 & (n = 0) \\ 1 + q + q^2 + \ldots + q^{n-1} & (n > 0) \end{cases}$$

so that

$$\underline{1}_n = n,$$

and

$$\underline{1}_n! = n!,$$

where

$$\underline{q}_n! = \underline{q}_n \underline{q}_{n-1} \cdots \underline{q}_1.$$

Some congruence properties and relationships with Bernoulli and Fibonacci numbers are explored. Some aspects of the notation and meaning of the Fermatian numbers are also outlined.

1. INTRODUCTION

One of the first variations of the q-binomial coefficients studied by Gauss and Cayley (Macmahon [21]) and Cauchy [10] were suggested by Fontené [15] who used an arbitrary sequence $\{A_n\}$ of real or complex numbers instead of the natural numbers. The ordinary binomial coefficients were the special case $A_n = n$ and the q-binomial coefficients for $A_n = q^n - 1$, the n-th Fermatian function of q.

Morgan Ward [26] independently rediscovered Fontené's generalized coefficients. Later Gould [16] developed some striking theorems for what he called the Fontené-Ward generalized binomial coefficients. He used

$$A_n = 1 + q + q^2 + \ldots + q^{n-1}, \tag{1.1}$$

although we shall designate this by $\underline{q}_n$ for subsequent notational convenience. More formally, we define the n-th reduced Fermatian number in terms of

$$\underline{q}_n = \begin{cases} -q^n \underline{q}_{-n} & (n<0) \\ 1 & (n=0) \\ 1+q+q^2+\ldots+q^{n-1} & (n>0) \end{cases} \tag{1.2}$$

so that

$$\underline{1}_n = n,$$

and

$$\underline{1}_n! = n!,$$

where

$$\underline{q}_n! = \underline{q}_n \underline{q}_{n-1} \cdots \underline{q}_1. \tag{1.3}$$

It is the purpose of this paper to examine some of the properties of these Fermatian numbers, which have been considered elsewhere by the present writer in another context [24].

Some examples of Fermatian numbers with various indices are displayed in Table 1.

$q\downarrow$ $n\rightarrow$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	1	3	7	15	31	63	127	255	511
3	1	4	13	40	121	364	1093	3280	9841
4	1	5	21	85	341	1365	5461	21845	87381
5	1	6	31	156	781	3906	19531	97656	488281
6	1	7	43	259	1555	9331	55987	335923	2015539
7	1	8	57	400	2801	19608	137257	960800	6725601
8	1	9	73	585	4681	37449	299593	2396745	19173961
9	1	10	91	820	7381	66430	597871	5380840	48427561

Table1: First Nine Fermatians of the First Nine Indices

2. CONGRUENCE PROPERTIES

We begin with some congruence properties (modulo $\underline{q}_n$).

$$\underline{q}_{n+1} \equiv 1 (\bmod \underline{q}_n). \tag{2.1}$$

Proof:

$$\begin{aligned} \underline{q}_{n+1} &= 1+q+q^2+\ldots+q^n \\ &= 1+q(1+q+\ldots+q^{n-1}) \\ &= 1+q\underline{q}_n \\ &\equiv 1 (\bmod \underline{q}_n). \end{aligned}$$

■

More generally,

$$\underline{q}_{2n-r} \equiv \underline{q}_{n-r} (\bmod \underline{q}_n). \tag{2.2}$$

Proof:

$$\begin{aligned}
\underline{q}_{2n-r} &= 1+q+\ldots+q^{n-r-1}+q^{n-r}+\ldots+q^{2n-r-1} \\
&= \underline{q}_{n-r}+q^{n-r}\underline{q}_{n} \\
&\equiv \underline{q}_{n-r}\ (\operatorname{mod} \underline{q}_{n}).
\end{aligned}$$

■

A corollary of this is that

$$\begin{bmatrix} 2n-r \\ n \end{bmatrix} \equiv 1 (\operatorname{mod} \underline{q}_{n}) \tag{2.3}$$

in which

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{\underline{q}_{n}!}{\underline{q}_{r}!\underline{q}_{n-r}} \tag{2.4}$$

is an analog of the ordinary binomial coefficient.

Proof:

$$\begin{aligned}
\begin{bmatrix} 2n-r \\ n \end{bmatrix} &= \frac{\underline{q}_{2n-r}!}{\underline{q}_{n}!\underline{q}_{n-r}!} \\
&= \frac{\underline{q}_{2n-r}\ldots\underline{q}_{2n-n+1}}{\underline{q}_{n-r}\ldots\underline{q}_{n-n+1}} \\
&\equiv 1 (\operatorname{mod} \underline{q}_{n}).
\end{aligned}$$

■

$$q\underline{q}_{n-1} \equiv -1 (\operatorname{mod} \underline{q}_{n}). \tag{2.5}$$

Proof:

$$\begin{aligned}
q\underline{q}_{n-1} &= q\left(1+q+\ldots+q^{n-2}\right) \\
&= q+q^{2}+\ldots+q^{n-1} \\
&= \underline{q}_{n}-1 \\
&\equiv -1 (\operatorname{mod} \underline{q}_{n}).
\end{aligned}$$

■

$$\underline{q}_{mr} = \underline{q}_{m}\underline{\left(q^{m}\right)}_{r}. \tag{2.6}$$

Proof:

$$\begin{aligned}
\underline{q}_{mr} &= \frac{1-q^{mr}}{1-q},\ (q \neq 1) \\
&= \frac{\left(1-q^{m}\right)\left(1-q^{mr}\right)}{(1-q)\left(1-q^{m}\right)} \\
&= \underline{q}_{m}\underline{\left(q^{m}\right)}_{r}.
\end{aligned}$$

■

3. CONNECTIONS WITH BERNOULLI AND FIBONACCI NUMBERS

$$\sum_{q=1}^{k-1} \underline{q}_n = \sum_{m=0}^{n-1} \frac{1}{m-r+1} \begin{bmatrix} m \\ r \end{bmatrix} k^{m-r+1} B_r \tag{3.1}$$

in which the B_r are ordinary Bernoulli numbers defined by the recurrence relation

$$B_n = \sum_{r=0}^{n} \binom{n}{r} B_r$$

with initial conditions $B_0 = 1,\ B_1 = -\frac{1}{2}$.

Proof of (3.1):

$$\begin{aligned}
\sum_{q=1}^{k-1} \underline{q}_n &= \sum_{q=1}^{k-1} \left(1 + q + q^2 + \ldots + q^{n-1}\right) \\
&= \sum_{m=0}^{n-1} \sum_{q=1}^{k-1} q^m \\
&= \sum_{m=0}^{n-1} \left(1^m + 2^m + \ldots + (k-1)^m\right) \\
&= \sum_{m=0}^{n-1} \sum_{r=0}^{m} \frac{1}{m-r+1} \begin{bmatrix} m \\ r \end{bmatrix} k^{m-r+1} B_r
\end{aligned}$$

by the Euler-Maclaurin sum formula [17]. ■

$$\underline{\left(\tfrac{\alpha}{\beta}\right)}_n + \underline{\left(\tfrac{\beta}{\alpha}\right)}_n = F_n L_{n-1} (-1)^{n-1}, \tag{3.2}$$

in which F_n, L_n are the Fibonacci and Lucas numbers respectively which can be expressed in their Bernoulli-Binet form [25] by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

$$L_n = \alpha^n + \beta^n,$$

where α,β are the roots of the characteristic equation of the corresponding second-order linear homogenous recurrence relation

$$x^2 - x - 1 = 0.$$

Proof of (3.2):

$$\begin{aligned}
F_n &= \left(\alpha^n - \beta^n\right) / (\alpha - \beta) \\
&= \alpha^{n-1} \left(1 - (\beta / \alpha)^n\right) / \left(1 - (\beta / \alpha)\right)
\end{aligned}$$

and so

$$\beta^{n-1}F_n = (-1)^{n-1}\underline{\left(\tfrac{\beta}{\alpha}\right)}_n .$$

Similarly,

$$\alpha^{n-1}F_n = (-1)^{n-1}\underline{\left(\tfrac{\alpha}{\beta}\right)}_n$$

Thus,

$$\begin{aligned}\underline{\left(\tfrac{\alpha}{\beta}\right)}_n + \underline{\left(\tfrac{\beta}{\alpha}\right)}_n &= (-1)^{n-1}F_n(\alpha^{n-1}+\beta^{n-1}) \\ &= (-1)^{n-1}F_n L_{n-1}.\end{aligned}$$

■

More generally, for Horadam's generalized sequence of numbers $\{w_n\}$ [19]

$$w_n = b\alpha^{n-1}\underline{\left(\tfrac{\beta}{\alpha}\right)}_n - aq\alpha^{n-2}\underline{\left(\tfrac{\beta}{\alpha}\right)}_{n-1}, \tag{3.3}$$

Where α, β are the roots of $x^2 - px + q = 0$.

Proof of (3.3):

$$\begin{aligned} w_n &= \frac{b-a\beta}{\alpha-\beta}\alpha^n + \frac{a\alpha-b}{\alpha-\beta}\beta^n \\ &= \left(b\left(\alpha^n-\beta^n\right) - a\alpha\beta\left(\alpha^{n-1}-\beta^{n-1}\right)\right)/\left(\alpha-\beta\right) \\ &= b\alpha^{n-1}\underline{\left(\tfrac{\beta}{\alpha}\right)}_n - aq\alpha^{n-2}\underline{\left(\tfrac{\beta}{\alpha}\right)}_{n-1}.\end{aligned}$$

■

4. COMBINATORIAL ASPECTS

The writer feels that $\underline{q}_n$ has some notational advantages. If

$$T_n = T_{n-1} + T_{n-2} + \ldots + T_{n-r}, \tag{4.1}$$

then T_n is the sum of the rising diagonals of the multinomial triangle generated by $\underline{q}_r^n$ [11,13,14]. The notational advantages of $\underline{q}_n$ can be further illustrated by reference to Hoggatt and Bicknell [18]. They prove that, for the general r-nomial triangle induced by the expansion of $\underline{q}_r^n$, (n=0,1,2,3,…), by letting the r-nomial triangle be left-justified and by taking sums from the left edge and jumping up p and over 1 entry until out of the triangle that

$$T_n = \sum_{k=0}^{\left\lfloor \frac{1}{2}n(r-1)\right\rfloor} \begin{Bmatrix} n-k \\ k \end{Bmatrix}_k, \tag{4.2}$$

where

$$\underline{q}_r^n = \sum_{j=0}^{n(r-1)} \begin{Bmatrix} n \\ j \end{Bmatrix}_r q^j, \tag{4.3}$$

and the r-nomial coefficient $\left\{ \begin{matrix} n \\ j \end{matrix} \right\}_r$ is the entry in the n-th row and the j-the column of the generalized Pascal triangle. Thus

$$\left(1 - x\underline{\left(x^p\right)}_r\right)^{-1} = \sum_{n=0}^{\infty}\left(x\underline{\left(x^p\right)}_r\right)^n$$
$$= \sum_{n=0}^{\infty}\left(\sum^{\left\lfloor \frac{n(r-1)}{p+1} \right\rfloor}\left\{ \begin{matrix} n-kp \\ k \end{matrix} \right\}_r\right)x^n,$$

which, when p=1, is a generating function for $\{T_n\}$ with suitable initial values. In this formula

$$\underline{\left(x^p\right)}_r = 1 + x^p + x^{2p} + ... + x^{p(r-1)}, \tag{4.4}$$

so that the notation is quite versatile.

Carlitz [6] has developed a generalization of Wilson's Theorem, but efforts to find such a generalization in terms of $\underline{q}_{p-1}!$ were unsuccessful. Its development would enable the construction of a Fermatian Staudt-Clausen Theorem [20].

Efforts to find physical descriptions of $\underline{q}_n$ have not yielded much success either. Possible approaches include the following. The enumerator for the number of r-permutations with repetition of n different things and no m consecutive things alike is denoted by

$$a_n^{(m)}(x) = \sum_{r=1}^{\infty} a_{n,r}^{(m)} x^n / n! \tag{4.5}$$

where $a_{n,r}^{(m)}$ are the numbers. This is due to Riordan [22] who has shown that

$$a_n^{(m)}(x) = \frac{n\underline{x}_{m-2}}{1-(n-1)x\underline{x}_{m-2}}, \tag{4.3}$$

which gives

$$\underline{x}_m = \left(1-(n-1)x\underline{x}_m\right)a_n^{(m+2)}(x). \tag{4.4}$$

It is interesting to note, that for m=3, it follows from Riordan's Problem 17(b) that

$$a_{n,r}^{(3)} = \frac{n}{n-1}F_{r+1}, \tag{4.5}$$

in terms of Fibonacci numbers. Chapter XVI of Dickson [12] is devoted to a description of the properties of various interpretations of q_n.

Riordan [23] has also shown that

$$\underline{q}_{n+1} = 1 + a_{n1}(q) \tag{4.6}$$

where $a_{nm}(q)$ is the enumerator of partitions with *m* parts, none greater than *n*, such that their Ferrer's graphs include an initial triangle of sides *n* and *m* (the graph of partition *m,m*-1,...,2,1).

Carlitz used $p_n(m)$ and $N_n(r)$ in formulae which can be adjusted to include $\underline{x}_i$, where $p_n(m)$ is the number of partitions of *m* into parts not exceeding *n*, and

$$N_n(r) = N(r; k_1, k_2, ..., k_n)$$

is the number of sequences of

$$\sigma = (a_1, a_2, ..., a_m)$$

with exactly *r* inversions where

$$m = k_1 + k_2 + ... + k_n.$$

It is known that

$$\prod_{i=1}^{n} (\underline{m}_i)^{-1} = (1-m)^n p_n(m)$$

and

$$\prod_{i=1}^{n} \underline{x}_i = \sum_{r=0}^{\frac{1}{2}n(n-1)} N_n(r) x^r.$$

The rising and falling factorials, Fermatians and Bernoulli numbers can all in fact be related by Carlitz' note on a theorem of Glaisher [3,4,5]: for

$$x^{\bar{p}} = x^{p-1} + A_1 x^{p-2} + ... + A_{p-1}$$

it can be shown that

$$\tfrac{1}{2} p(p-r) A_{r-1} \equiv A_r \ (\text{mod}\ \underline{p}_5)$$

and

$$\tfrac{1}{2} p(p-r) A_{r-1} - A_r \equiv r^3 B_{r-3} p^4 / 24(r-3) \ (\text{mod}\ \underline{p}_6).$$

The $\underline{q}_n$ may also be considered in their role as cyclotomic polynomials, a topic to which Carlitz devoted a number of papers.

5. CONCLUDING COMMENTS

Carlitz and Moser [9] examined some of the Fermatian properties by giving all the possible factorizations of $\underline{x}_n$ into its product of *C*-polynomials over the field of rational numbers, where the *C*-polynomial of *A* is defined by

$$A(x) = x^{a_1} + x^{a_2} + ... + x^{a_k} \tag{5.1}$$

for

$$A = \{a_1, a_2, ..., a_k\},$$

an ordered set of non-negative integers. A particularly interesting result of Carlitz and Moser is that if *f(n)* denotes the number of factorizations,

$$\underline{x}_n = A(x)B(x),$$

where *A(x),B(x)* are *C*-polynomials, then

$$2f(n) = \sum_{d|n} f(d).$$

Elsewhere [7] Carlitz proved, that, for the quotient

$$Q_n = \frac{((p-1)n)!}{(n!)^{p-1}}, \tag{5.2}$$

the highest power of the prime *p*, that divides Q_n is 0 when $n = \underline{p}_j$, and is (*aj-j*) when $n = a\underline{p}_j$. Carlitz [2] has also used $\underline{q}_n$ in the development of *q*-Bernoulli numbers and polynomials. Carlitz used the notation [*x*] such that

$$[x] = (q^x - 1)/(q-1), \tag{5.3}$$

but as [*x*] is also used for the greatest integer function and Carlitz himself [1] also used [*k*] to mean

$$[k] = x^{p^{nk}} - x,$$

it was felt that the notation used in this paper is less confusing and more suggestive.

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