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ON 25-TH AND 26-TH SMARANDACHE'S PROBLEMS

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The 25-th and 26-th problems from [1] (see also 30-th and 31-st problems from [2]) are the following:

25. Smarandache's cube free sieve:

$$2,3,4,5,6,7,9,10,11,12,13,14,15,17,18,19,20,21,22,23,25,26,28,29,$$

$$30,31,33,34,35,36,37,38,39,41,42,43,44,45,46,47,49,50,51,52,53,55,$$

$$57,58,59,60,61,62,63,65,66,67,68,69,70,71,73,...$$

Definition: from the set of natural numbers (except 0 and 1):
– take off all multiples of 2^3 *(i.e. 8,16,24,32,40,...)*
– take off all multiples of 3^3
– take off all multiples of 5^3
... and so on (take off all multiples of all cubic primes).
(One obtains all cube free numbers.)

26. Smarandache's m-power free sieve:

Definition: from the set of natural numbers (except 0 and 1) take off all multiples of 2^m, *afterwards all multiples of* 3^m *... and so on (take off all multiples of all m-power primes,* $m \geq 2$).
(One obtains all m-power free numbers.)

Here we shall introduce the solutions of both these problems.

For every natural number m we denote the increasing sequence $a_1^{(m)}, a_2^{(m)}, a_3^{(m)}, ...$ of all m-power free numbers by $\overline{m}$. Then we have

$$\emptyset \equiv \overline{1} \subset \overline{2} ... \subset \overline{(m-1)} \subset \overline{m} \subset \overline{(m+1)} \subset ...$$

Also, for $m \geq 2$ we have

$$\overline{m} = \bigcup_{k=1}^{m-1} (\overline{2})^k$$

where

$$(\overline{2})^k = \{x \mid (\exists x_1, ..., x_k \in \overline{2})(x = x_1.x_2. \ldots .x_k)$$

for each natural number $k \geq 1$.

Let us consider $\overline{m}$ an infinite sequence for $m = 2, 3, \ldots$. Then $\overline{2}$ is a subsequence of $\overline{m}$. Therefore, the inequality

$$a_n^{(m)} \leq a_n^{(2)}$$

holds for $n = 1, 2, 3, \ldots$.

Let $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$ be the sequence of all primes. It is obvious that this sequence is a subsequence of $\overline{2}$. Hence, the inequality

$$a_n^{(2)} \leq p_n$$

holds for $n = 1, 2, 3, \ldots$. But it is well known that

$$p_n \leq \lambda(n) \equiv [\frac{n^2 + 3n + 4}{4}] \tag{1}$$

(see [3]). Therefore, for any $m \geq 2$ and $n = 1, 2, 3, \ldots$ we have

$$a_n^{(m)} \leq a_n^{(2)} \leq \lambda(n). \tag{2}$$

Further, we will find an explicit formula for $a_n^{(m)}$ when $m \geq 2$ is fixed.

Let for any real x

$$sg(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

We define

$$\varepsilon_m(k) = \begin{cases} 1, & k \in \overline{m} \\ 0, & k \notin \overline{m} \end{cases}.$$

Hence,

$$\pi_{\overline{m}}(n) = \sum_{k=2}^{n} \varepsilon_m(k), \tag{3}$$

where $\pi_{\overline{m}}(n)$ is the number of terms of set $\overline{m}$, which are not greater than n. Using the relation

$$\varepsilon_m(k) = sg\Big(\prod_{\substack{p \mid k \\ p \text{ is prime}}} [\frac{m-1}{ord_p k}]\Big)$$

we rewrite (3) in the explicit form

$$\pi_{\overline{m}}(n) = \sum_{k=2}^{n} sg(\prod_{\substack{p|k \\ p \text{ is prime}}} [\frac{m-1}{ord_p k}]). \tag{4}$$

Then, using formulae (1')-(3') from [4] (which are the universal formulae for the n-th term of an arbitrary increasing sequence of natural numbers), and (2), with $\lambda(n)$ from (1), we obtain

$$a_n^{(m)} = \sum_{k=0}^{\lambda(n)} [\frac{1}{1+[\frac{\pi_{\overline{m}}(k)}{n}]}]; \tag{5}$$

$$a_n^{(m)} = -2\sum_{k=0}^{\lambda(n)} \zeta(-2[\frac{\pi_{\overline{m}}(k)}{n}]); \tag{6}$$

(a representation using Riemann's function ζ)

$$a_n^{(m)} = \sum_{k=0}^{\lambda(n)} \frac{1}{\Gamma(1-[\frac{\pi_{\overline{m}}(k)}{n}])}, \tag{7}$$

(a representation using Euler's function Γ).

We note that (5)-(7) are explicit formulae, because of (4).

Thus, the 26-th Smarandache's problem is solved and for $m = 3$ the 25-th Smarandache's problem is solved, too.

For $m = 2$ we have the representation

$$\varepsilon_2(k) = |\mu(k)|$$

(here μ is the Möbius function);

$$|\mu(k)| = [\frac{2^{\omega(k)}}{\tau(k)}],$$

where $\omega(k)$ denotes the number of all different prime divisors of k and

$$\tau(k) = \sum_{d|k} 1.$$

Hence,

$$\pi_{\overline{2}}(n) = \sum_{k=2}^{n} |\mu(k)| = \sum_{k=2}^{n} [\frac{2^{\omega(k)}}{\tau(k)}].$$

The following problems are interesting.

Problem 1: Is there a constant $C > 1$, such that $\lambda(n) \leq C.n$?

Problem 2: Is $C \leq 2$?

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Below we give the main explicit representation of function $\pi_{\overline{m}}(n)$, that takes part in formulae (5)-(7). In this way we find the main explicit representation for $a_n^{(m)}$, that is based on formulae (5)-(7), too.

Theorem: Function $\pi_{\overline{m}}(n)$ allows representation

$$\pi_{\overline{m}}(n) = n - 1 + \sum_{s \in \overline{2} \cap \{2,3,\ldots,[\sqrt[m]{n}]\}} (-1)^{\omega(s)}.[\frac{n}{s^m}]. \tag{8}$$

Proof: First, we shall note that the sum in the right hand of (8) is over only these natural numbers s, smaller than $[\sqrt[m]{n}]$, for which $s \in \overline{2}$, i.e., over these natural numbers s for which $\mu(s) \neq 0$.

Let $\{b_n^{(m)}\}_{m=1}^{\infty}$ be the sequence defined by

$$b_1^{(m)} = 1, \; b_n^{(m)} = a_{n-1}^{(m)} \text{ for } n \geq 2. \tag{9}$$

We shall denote this sequence by m^*.

Let $\pi_{m^*}(n)$ denote the number of terms of m^*, which are not greater than n. Then we have relation

$$\pi_{\overline{m}}(n) = \pi_{m^*}(n) - 1, \tag{10}$$

because of (9).

Let $g^{(m)}(k)$ be the function given by

$$g^{(m)}(k) = \begin{cases} 1, & k \in m^* \\ 0, & k \notin m^* \end{cases}. \tag{11}$$

Then $g^{(m)}(k)$ is a multiplicative function with respect to k, i.e., $g^{(m)}(1) = 1$ and for every two natural numbers a and b, such that $(a, b) = 1$, relation

$$g^{(m)}(a.b) = g^{(m)}(a).g^{(m)}(b)$$

holds.

Let function $f^{(m)}(k)$ be introduced by

$$f^{(m)}(k) = \sum_{d/k} \mu(\frac{k}{d}) g^{(m)}(d). \tag{12}$$

Using (12) for $k = p^\alpha$, where p is an arbitrary prime and α is an arbitrary natural number, we obtain

$$f^{(m)}(p^\alpha) = g^{(m)}(p^\alpha) - g^{(m)}(p^{\alpha-1}).$$

Hence,

$$f^{(m)}(p^\alpha) = \begin{cases} 0, & \alpha < m \\ -1, & \alpha = m \\ 0, & \alpha > m \end{cases},$$

because of (11).

Hence, $f^{(m)}(1) = 1$ and for $k \geq 2$ we have

$$f^{(m)}(k) = \begin{cases} (-1)^{\omega(k)}, & \text{if } k \text{ is an } m\text{-power natural number and } k \in \overline{2} \\ 0, & \text{otherwise} \end{cases}, \tag{13}$$

since $f^{(m)}(k)$ is a multiplicative function with respect to k, because of (12).

Using the Möbius inversion formula, equality (12) yields

$$g^{(m)}(k) = \sum_{d/k} f^{(m)}(d). \tag{14}$$

Now, we use (14) and the obvious representation

$$\pi_{m^*}(n) = \sum_{k=1}^{n} g^{(m)}(k) \tag{15}$$

in order to obtain

$$\pi_{m^*}(n) = \sum_{k=1}^{n} \sum_{d/k} f^{(m)}(d). \tag{16}$$

Then (16) and the identity

$$\sum_{k=1}^{n} \sum_{d/k} f^{(m)}(d) = \sum_{k=1}^{n} f^{(m)}(k).[\frac{n}{k}] \tag{17}$$

both yield

$$\pi_{m^*}(n) = \sum_{k=1}^{n} f^{(m)}(k).[\frac{n}{k}]. \tag{18}$$

From (13) and (18) we obtain (8), because of (10) and the fact that $f^{(m)}(1) = 1$. The theorem is proved.

Finally, we must note that some authors call function $(-1)^{\omega(s)}$ unitary analogue of the Möbius function $\mu(s)$ and denote this function by $\mu^*(s)$ (see [5, 6]). So, if we agree to use the last notation, we may rewrite formula (8) in the form

$$\pi_{\overline{m}}(n) = n - 1 + \sum_{s \in \overline{2} \cap \{2,3,\ldots,[\sqrt[m]{n}]\}} \mu^*(s).[\frac{n}{s^m}].$$

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