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On the Hadamard product of GCD and LCM matrices associated with semi-multiplicative functions

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Abstract

Let $S = \{x_1, x_2,\ldots,x_n\}$ be an ordered set of distinct positive integers with $x_1<x_2< \ldots <x_n$ $(n>1)$. We provide some properties for the $n\times n$ matrix $(G)_f \circ [L]_f$ on S, where $(G)_f$ denotes the $n\times n$ GCD matrix on S associated with f, $[L]_f$ denotes the $n\times n$ LCM matrix on S associated with f and o denotes the Hadamard product.

Keywords: Hadamard product, GCD matrix, LCM matrix, semi-multiplicative function.

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1. Introduction

Let $S = \{x_1, x_2,\ldots,x_n\}$ be an ordered set of distinct positive integers with $x_1< x_2 <\ldots< x_n$ $(n>1)$ and let f be an arithmetical function. The $n\times n$ matrix $(G)_f$ is called the GCD matrix on S associated with f, where the i,j -entry is f evaluated at the greatest common divisor of x_i and x_j, that is, the i,j -entry of $(G)_f$ is $f((x_i,x_j))$. The $n\times n$ matrix $[L]_f$ is called the LCM matrix on S associated with f, where the i,j -entry is f evaluated at the least common multiple of x_i and x_j, that is, the i,j -entry of $[L]_f$ is $f([x_i,x_j])$.

H. J. S. Smith [7] calculated the determinant of the GCD and LCM matrices under certain conditions. Since Smith a large number of results on GCD and LCM matrices have been presented in the literature. For general accounts see e.g. [2], [3] and [5].

Beslin [1] studies the determinant of the LCM matrix and the reciprocal GCD matrix on S, that is, the determinant of the matrices, whose i,j -entries are $[x_i, x_j]$ and $1/(x_i, x_j)$, respectively. Tuglu and Tasci [8] study the determinant, the trace and the inverse of the Hadamard product of the LCM matrix and the reciprocal GCD matrix on S.

In this paper we study basic properties of the Hadamard product $(G)_f$ o $[L]_f$ of the GCD and LCM matrix on S associated with a semi-multiplicative function f. We show that $\det((G)_f \text{ o } [L]_f)=0$, determine the eigenvalues and eigenvectors of $(G)_f$ o $[L]_f$ and evaluate certain norms of $(G)_f$ o $[L]_f$.

2. Results

Let f be a real-valued arithmetical function. Then f is said to be semi-multiplicative if $f(r)f(s)=f((r, s))f([r, s])$ for all positive integers r and s. Semi-multiplicative functions f with $f(1)\neq 0$ are quasi-multiplicative functions, and quasi-multiplicative functions f with $f(1)=1$ are the usual multiplicative functions. Multiplicative functions f with $f(p^e)=f(p)^e$ for all prime powers p^e are completely multiplicative functions. See [3, 6].

Throughout this section let $S = \{ x_1, x_2,\ldots,x_n \}$ be an ordered set of distinct positive integers with $x_1< x_2 <\ldots< x_n$ $(n>1)$ and let f be a semi-multiplicative function.

The Hadamard product of the $n\times n$ matrix $(G)_f$ and the $n\times n$ matrix $[L]_f$ on S is defined as $(G)_f \text{ o } [L]_f = (f((x_i, x_j))f([x_i, x_j]))$. Since f is a semi-multiplicative function, we have

$$(G)_f \text{ o } [L]_f = (f(x_i)f(x_j)). \qquad (2.1)$$

Note that the $n\times n$ matrix $(G)_f$ o $[L]_f$ on S is symmetric.

Theorem 2.1. $\det((G)_f \text{ o } [L]_f) = 0$.

Proof. From (2.1) we obtain

$$\det((G)_f \circ [L]_f)) = f(x_1)\, f(x_2) \ldots f(x_n) \det \begin{vmatrix} f(x_1) & f(x_1) & . & . & . & f(x_1) \\ f(x_2) & f(x_2) & . & . & . & f(x_2) \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ f(x_n) & f(x_n) & . & . & . & f(x_n) \end{vmatrix}.$$

Thus $\det((G)_f \circ [L]_f) = 0$. This completes the proof.

Corollary. The $n\times n$ matrix $(G)_f \circ [L]_f$ on S is always singular.

Theorem 2.2. Let f be a semi-multiplicative function with $f(r) \neq 0$ for all r. The eigenvalues of the $n\times n$ matrix $(G)_f \circ [L]_f$ on S are

$$\lambda_1 = \mathrm{tr}\,((G)_f \circ [L]_f),\ \lambda_{2,3,\ldots,n} = \det ((G)_f \circ [L]_f)=0.$$

Proof. Let λ_i, i=1, 2,…, n, denote the eigenvalues of the $n\times n$ matrix $(G)_f \circ [L]_f$. Let $\mathrm{alg}(\lambda_i)$ denote the algebraic multiplicity of λ_i, that is, $\mathrm{alg}(\lambda_i)$ is the multiplicity of λ_i as a zero of the characteristic polynomial of $(G)_f \circ [L]_f$, and let $\mathrm{geom}(\lambda_i)$ denote the geometric multiplicity of λ_i, that is, $\mathrm{geom}(\lambda_i)$ is the dimension of the eigenspace of $(G)_f \circ [L]_f$ corresponding to the eigenvalue λ_i (see [4]). Since the matrix $(G)_f \circ [L]_f$ is symmetric, it is diagonalizable. Thus $\mathrm{alg}(\lambda_i) = \mathrm{geom}(\lambda_i)$ for i=1, 2,…, n. Since the matrix $(G)_f \circ [L]_f$ is singular, $\lambda_i=0$ for some i=1, 2,…, n, say $\lambda_n=0$. Then $\mathrm{alg}(\lambda_n) = \mathrm{geom}(\lambda_n) = \mathrm{geom}(0) = n-\mathrm{rank}((G)_f \circ [L]_f)=n-1$. This means that the multiplicity of λ_n as a zero of the characteristic polynomial is equal to n-1. We thus may denote $\lambda_2=\lambda_3=\ldots=\lambda_n=0$. Since the sum of eigenvalues is equal to $\mathrm{tr}((G)_f \circ [L]_f)$, we have $\lambda_1= \mathrm{tr}((G)_f \circ [L]_f)$. This completes the proof.

Theorem 2.3. Let f be a semi-multiplicative function with $f(r) \neq 0$ for all r. Then eigenvectors corresponding to the eigenvalues of the $n\times n$ matrix $(G)_f \circ [L]_f$ on S are

$$y_1 = \begin{bmatrix} \frac{f(x_1)}{f(x_n)} \\ . \\ . \\ . \\ \frac{f(x_{n-1})}{f(x_n)} \\ 1 \end{bmatrix}, y_2 = \begin{bmatrix} -\frac{f(x_2)}{f(x_1)} \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, y_3 = \begin{bmatrix} -\frac{f(x_3)}{f(x_1)} \\ 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}, y_4 = \begin{bmatrix} -\frac{f(x_4)}{f(x_1)} \\ 0 \\ 0 \\ 1 \\ 0 \\ . \\ 0 \end{bmatrix}, \ldots, y_n = \begin{bmatrix} -\frac{f(x_n)}{f(x_1)} \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}.$$

Proof. If λ_i is an eigenvalue of the $n \times n$ matrix $(G)_f$ o $[L]_f$, the correspondings eigenvectors Y_i are the solutions of

$$(\lambda_i I - (G)_f \, o[L]_f)Y_i = 0. \tag{2.2}$$

We first calculate the eigenvector corresponding to λ_1 = tr $((G)_f$ o $[L]_f)$. From (2.2) ,

$$\left((f(x_1)^2 + f(x_2)^2 + \ldots + f(x_n)^2) I - (G)_f \, o[L]_f\right)Y_1 = 0. \tag{2.3}$$

If we denote unknown vectors Y_1 by $[y_1, y_2, \ldots, y_n]^T$, then (2.3) becomes

$$\begin{bmatrix} f(x_2)^2+...+f(x_n)^2 & -f(x_1)f(x_2) & ... & -f(x_1)f(x_n) \\ -f(x_2)f(x_1) & f(x_1)^2+f(x_3)^2+...+f(x_n)^2 & ... & -f(x_2)f(x_n) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ -f(x_n)f(x_1) & -f(x_n)f(x_2) & ... & f(x_1)^2+f(x_2)^2+...+f(x_{n-1})^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

By using elementary row operations, the coefficient matrix of this homogeneous system becomes

$$\begin{bmatrix} 1 & \frac{-f(x_1)f(x_2)}{f(x_2)^2+...+f(x_n)^2} & ... & \frac{-f(x_1)f(x_p)}{f(x_2)^2+...+f(x_n)^2} & ... & \frac{-f(x_1)f(x_{n-1})}{f(x_2)^2+...+f(x_n)^2} & \frac{-f(x_1)f(x_n)}{f(x_2)^2+...+f(x_n)^2} \\ 0 & 1 & ... & \frac{-f(x_2)f(x_p)}{f(x_3)^2+...+f(x_n)^2} & ... & \frac{-f(x_2)f(x_{n-1})}{f(x_3)^2+...+f(x_n)^2} & \frac{-f(x_2)f(x_n)}{f(x_3)^2+...+f(x_n)^2} \\ . & . & ... & . & ... & . & . \\ 0 & 0 & ... & 1 & ... & \frac{-f(x_p)f(x_{n-1})}{f(x_{p+1})^2+...+f(x_n)^2} & \frac{-f(x_p)f(x_n)}{f(x_{p+1})^2+...+f(x_n)^2} \\ . & . & ... & . & ... & . & . \\ 0 & 0 & ... & 0 & ... & 1 & -\frac{f(x_{n-1})}{f(x_n)} \\ 0 & 0 & ... & 0 & ... & 1 & -\frac{f(x_{n-1})}{f(x_n)} \end{bmatrix}$$

Since the rank of the coefficient matrix of this homogeneous system is equal to n-1 , there exist infinitely many solutions dependent on one parameter. The solution to this set of equations is

$$y_1 = \frac{f(x_1)}{f(x_n)} t \ , \ y_2 = \frac{f(x_2)}{f(x_n)} t \ , \dots, y_{n-1} = \frac{f(x_{n-1})}{f(x_n)} t \ , y_n = t \ ,$$

where t is arbitrary. In this case, linearly independent eigenvector corresponding to $\lambda_1 = \operatorname{tr}((G)_f \circ [L]_f)$ is equal to

$$\left[\frac{f(x_1)}{f(x_n)}, \frac{f(x_2)}{f(x_n)}, \dots, \frac{f(x_{n-1})}{f(x_n)}, 1 \right]^T .$$

Now we calculate the eigenvectors corresponding to $\lambda_{2,3,\dots,n} = 0$. From (2.2),

$$\begin{bmatrix} -f(x_1)^2 & -f(x_1)f(x_2) & \dots & -f(x_1)f(x_{n-1}) & -f(x_1)f(x_n) \\ -f(x_2)f(x_1) & -f(x_2)^2 & \dots & -f(x_2)f(x_{n-1}) & -f(x_2)f(x_n) \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ -f(x_n)f(x_1) & -f(x_n)f(x_2) & \dots & -f(x_n)f(x_{n-1}) & -f(x_n)^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}.$$

By using elementary row operations, the coefficient matrix of this homogeneous system becomes

$$\begin{bmatrix} -f(x_1)^2 & -f(x_1)f(x_2) & \dots & -f(x_1)f(x_{n-1}) & -f(x_1)f(x_n) \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ . & . & & . & . \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Since the rank of the coefficient matrix of this homogeneous system is equal to 1, there exist infinitely many solutions dependent on n-1 parameters. The solution to this set of equations is

$$y_1 = -\frac{f(x_2)}{f(x_1)} t_2 - \frac{f(x_3)}{f(x_1)} t_3 - \dots - \frac{f(x_n)}{f(x_1)} t_n, \ y_2 = t_2 \ , y_3 = t_3, \dots, y_n = t_n,$$

where t_2, t_3,…, t_n are arbitrary. In this case, linearly independent eigenvectors corresponding to $\lambda_{2,3,\dots,n} = 0$ are

$$\begin{bmatrix} -\frac{f(x_2)}{f(x_1)} \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{f(x_3)}{f(x_1)} \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{f(x_4)}{f(x_1)} \\ 0 \\ 0 \\ 1 \\ 0 \\ \cdot \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} -\frac{f(x_n)}{f(x_1)} \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}.$$

This completes the proof.

Remark Since the matrix $(G)_f$ o $[L]_f$ is symmetric, it is diagonalizable. In view of Theorems 2.2 and 2.3 we write

$$P = \begin{bmatrix} \frac{f(x_1)}{f(x_n)} & -\frac{f(x_2)}{f(x_1)} & -\frac{f(x_3)}{f(x_1)} & \cdots & -\frac{f(x_{n-1})}{f(x_1)} & -\frac{f(x_n)}{f(x_1)} \\ \frac{f(x_2)}{f(x_n)} & 1 & 0 & \cdots & 0 & 0 \\ \cdot & 0 & 1 & \cdots & 0 & 0 \\ & \cdot & 0 & & \cdot & \cdot \\ \cdot & & & \cdots & & \\ & \cdot & \cdot & & \cdot & \cdot \\ \frac{f(x_{n-1})}{f(x_n)} & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

to obtain

$$P^{-1}\,((G)_f \text{ o } [L]_f)\,P = \operatorname{diag}(f(x_1)^2 + f(x_2)^2 + \ldots + f(x_n)^2, 0, 0, \ldots, 0).$$

Definition [4] Let M_n denote the class of complex n×n matrices. The maximum column sum matrix norm on M_n is defined by

$$|||A|||_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} | a_{ij} |$$

and the maximum row sum matrix norm on M_n is defined by

$$|||A|||_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} | a_{ij} | .$$

The ℓ_1 norm on M_n is defined by

$$\|A\|_1 = \sum_{i,j=1}^{n} | a_{ij} |$$

and the Euclidean norm or ℓ_2 norm on M_n is defined by

$$\|A\|_2 = \left(\sum_{i,j=1}^{n} | a_{ij} |^2 \right)^{1/2} .$$

Theorem 2.4

(i) $\| (G)_f \circ [L]_f \|_1 = \| (G)_f \circ [L] \|_\infty = | f(x_n) | \left(| f(x_1) | + | f(x_2) | + \ldots + | f(x_n) | \right)$

if $|f|$ is increasing.

(ii) $\| (G)_f \circ [L]_f \|_1 = \left(f(x_1) + f(x_2) + \ldots + f(x_n) \right)^2$.

(iii) $\| (G)_f \circ [L]_f \|_2 = tr\left((G)_f \, o [L]_f \right)$.

Theorem 2.4 follows easily from the definition of the norms.

Remark. The results of this paper can be generalized as follows. Let (P, ≤) be a lattice. Let $S = \{ x_1, x_2, \ldots, x_n \}$ be a subset of P, and let f be a real-valued function on P. Then the n×n matrix $(G)_f = (f(x_i \wedge x_j))$ is called the meet matrix on S associated with f and the n×n matrix $[L]_f = (f(x_i \vee x_j))$ is called the join matrix on S associated with f. If $(P, \leq) = (\mathbf{Z}^+, |)$, then meet and join matrices, respectively, become GCD and LCM matrices, and if $(P, \leq) = (\mathbf{Z}^+, ||)$, then meet and join matrices, respectively, become GCUD and LCUM matrices. See [3, 5].

We say that f is a semi-multiplicative function if $f(x)f(y) = f(x \wedge y)f(x \vee y)$ for all $x, y \in P$. See [5]. If f is a semi-multiplicative function, then the Haramard product of the meet matrix $(G)_f$ and the join matrix $[L]_f$ is given as $(G)_f \; o \; [L]_f = (f(x_i \wedge x_j) f(x_i \vee x_j)) = (f(x_i) f(x_j))$. Thus our results hold for the Haramard product of meet and join matrices.

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