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q-VOLKENBORN INTEGRATION, II

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ABSTRACT. By using q-Volkenborn integration, the multiple Changhee q-Bernoulli numbers which are an interesting analogue of Barnes' multiple Bernoulli numbers were constructed in [4]. The object of this paper is to define the extension of multiple Changhee q-Bernoulli numbers and to give the new explicit formulas which are related to these numbers.

§1. INTRODUCTION

Let $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will be denoted by the ring of integers, the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$ and let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number or a p-adic number. If $q \in \mathbb{C}_p$, then we normally assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

In this paper, we use the notation:

$$[x] = [x : q] = \frac{1-q^x}{1-q}.$$

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Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case.

Let d be a fixed integer and let p be a fixed prime number. We set

$$X = \varprojlim_{N}(\mathbb{Z}/dp^N\mathbb{Z}),$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$
$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$, cf. [2,3,4,5,6,7,8].

For any positive integer N, we set

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]}$$

and this can be extended to a distribution on X. This distribution yields an integral for each non-negative integer m:

$$\int_{\mathbb{Z}_p} [x]^m d\mu_q(x) = \int_X [a]^m d\mu_q(a) = \frac{1}{(1-q)^m}\sum_{i=0}^{m}\binom{m}{i}(-1)^i\frac{i+1}{[i+1]}, \text{ cf. [5] .}$$

The multiple Barnes' Bernoulli polynomials were defined by

$$\left(\prod_{j=1}^{r}\frac{w_j}{e^{w_j t}-1}\right)t^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x|w_1, w_2, \cdots, w_r)\frac{t^n}{n!}, \tag{1}$$

for each $w_j > 0, 0 < t < 1$, cf. [1].

The numbers $B_n^{(r)}(w_1, w_2, \cdots, w_r) = B_n^{(r)}(0|w_1, w_2, \cdots, w_r)$ are called the multiple Barnes' numbers.

Throughout this paper, we assume that $\alpha_1, \cdots, \alpha_k$ are taken in the positive integers and let $w \in \mathbb{Z}_p$. Now, we can rewrite the multiple Changhee q-Bernoulli numbers, polynomials as follows, cf. [4]:

$$\beta_n^{(r)}(w, q|\alpha_1, \alpha_2, \cdots, \alpha_r) = \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r \text{ times}} [w + \alpha_1 x_1 + \cdots + \alpha_r x_r]^n d\mu_q(x_1)\cdots d\mu_q(x_r),$$

and

$$\beta_n^{(r)}(q|\alpha_1, \alpha_2, \cdots, \alpha_r) = \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^n d\mu_q(x_1)\cdots d\mu_q(x_r).$$

In this paper, we construct the numbers $\beta_n^{(h,r)}(w, q|\alpha_1, \alpha_2, \cdots, \alpha_r)$ for $h \in \mathbb{Z}$ which reduce the multiple Barnes' Bernoulli numbers $B_n^{(r)}(\alpha_1, \alpha_2, \cdots, \alpha_r)$ as $q \to 1$. Also, we give the new explicit formulas which are related to these numbers.

2. An extension of Changhee q-Bernoulli numbers

For $h \in \mathbb{Z}$, we define the extension of Changhee q-Bernoulli polynomials, numbers as follows:

$$\begin{aligned} &\beta_n^{(h,r)}(w, q|\alpha_1, \alpha_2, \cdots, \alpha_r) \\ &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r \text{ times}} q^{\sum_{i=1}^{r}(h-i)x_i}[w + \alpha_1 x_1 + \cdots + \alpha_r x_r]^n d\mu_q(x_1)\cdots d\mu_q(x_r), \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\beta_n^{(h,r)}(q|\alpha_1, \alpha_2, \cdots, \alpha_r) \\ &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^n q^{\sum_{i=1}^{r}(h-i)x_i} d\mu_q(x_1)\cdots d\mu_q(x_r). \end{aligned} \tag{3}$$

These can be written as

$$\beta_n^{(h,r)}(w, q|\alpha_1, \alpha_2, \cdots, \alpha_r) = \sum_{j=0}^{n}\binom{n}{j} q^{wj}\beta_j^{(h,r)}(q|\alpha_1, \alpha_2, \cdots, \alpha_r)[w]^{n-j}.$$

By (3), we have

$$\begin{aligned} &\beta_n^{(h,r)}(q|\alpha_1, \alpha_2, \cdots, \alpha_r) \\ &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^n q^{\sum_{i=1}^{r}(h-i)x_i} d\mu_q(x_1)\cdots d\mu_q(x_r) \\ &= \frac{1}{(1-q)^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^j \frac{(j\alpha_1 + h)(j\alpha_2 + h - 1)\cdots(j\alpha_r + h - r + 1)}{[j\alpha_1 + h][j\alpha_2 + h - 1]\cdots[j\alpha_r + h - r + 1]}. \end{aligned} \tag{4}$$

Therefore we obtain the following :

Theorem 1. *For any $n \geq 0$, we have*

$$\beta_n^{(h,r)}(w, q|\alpha_1, \alpha_2, \cdots, \alpha_r) = \frac{1}{(1-q)^n} \sum_{j=0}^{n} \binom{n}{j} (-q^w)^j \prod_{l=1}^{r} \left(\frac{j\alpha_l + h - l + 1}{[j\alpha_l + h - l + 1]} \right).$$

Remark 1. Note that

$$\beta_0^{(2,1)}(q|1) = \frac{2}{[2]}, \quad \beta_1^{(2,1)}(q|1) = \frac{2q+1}{[2][3]}, \quad \beta_2^{(2,1)}(q|1) = \frac{2q^2}{[3][4]},$$

$$\beta_3^{(2,1)}(q|1) = -\frac{q^2(q-1)(2[3]+q)}{[3][4][5]}, \cdots,$$

$$\beta_0^{(h,1)}(q|1) = \frac{h}{[h]},$$

$$\beta_1^{(h,1)}(q|1) = -\frac{(1+q+\cdots+q^{h-1}) + q(1+q+\cdots+q^{h-2}) + \cdots + q^{h-1}}{[h][h+1]} \cdots$$

$$\beta_0^{(2,2)}(q|1,1) = \frac{2!}{[2][1]}, \quad \beta_1^{(2,2)}(q|1,1) = -\frac{2(q+2)}{[2][3]},$$

$$\beta_2^{(2,2)}(q|1,1) = -\frac{2((q-1)^2+5q)}{[3][4]}, \cdots,$$

$$\beta_0^{(r,r)}(q|\underbrace{1, \cdots, 1}_{r \text{ times}}) = -\frac{r!}{[r][r-1]\cdots[2][1]}.$$

Remark 2. By the definition of $\beta_n^{(h,r)}(q|\alpha_1, \alpha_2, \cdots, \alpha_r)$, we note that

$$\beta_n^{(h,r)}(0, q|\alpha_1, \alpha_2, \cdots, \alpha_r) = \beta_n^{(h,r)}(q|\alpha_1, \alpha_2, \cdots, \alpha_r).$$

By (2),(3), it is easy to see that

$$\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^n q^{\sum_{i=1}^r (h-i)x_i} d\mu_q(x_1) \cdots d\mu_q(x_r) \\
&= (q-1) \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^{n+1} q^{\sum_{i=1}^r (h-\alpha_i-i)x_i} d\mu_q(x_1) \cdots d\mu_q(x_r) \\
&+ \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\alpha_1 x_1 + \cdots + \alpha_r x_r]^n q^{\sum_{i=1}^r (h-\alpha_i-i)x_i} d\mu_q(x_1) \cdots d\mu_q(x_r).
\end{aligned}$$

Thus we have

$$\beta_m^{(h,r)}(q|\underbrace{1,1,\cdots,1}_{r \text{ times}}) = (q-1)\beta_m^{(h-1,r)}(q|\underbrace{1,1,\cdots,1}_{r \text{ times}}) + \beta_m^{(h-1,r)}(q|\underbrace{1,1,\cdots,1}_{r \text{ times}}).$$

It is easy to see that

$$\begin{aligned}
&\underbrace{\int_X \cdots \int_X}_{r \text{ times}} [w+\alpha_1 x_1+\cdots+\alpha_r x_r]^n q^{\sum_{i=1}^r (h-i)x_i} d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= [d]^{n-r} \sum_{i_1,i_2,\cdots i_r=0}^{d-1} q^{(i_1+\cdots i_r)h-i_2-2i_3-\cdots-(r-1)i_r} \\
&\times \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [\frac{w+\alpha_1 i_1+\cdots+\alpha_r i_r}{d}+\alpha_1 x_1+\cdots+\alpha_r x_r; q^d]^n \\
&\times q^{x_1(h-1)d+\cdot+x_r(h-r)d} d\mu_{q^d}(x_1)\cdots d\mu_{q^d}(x_r).
\end{aligned} \tag{5}$$

From (3), (5), we have the following:

Theorem 2. *For any positive integer n, we have*

$$\begin{aligned}
\beta_n^{(h,r)}(w,q|\alpha_1,\alpha_2,\cdots,\alpha_r) = [d]^{n-r} &\sum_{i_1,i_2,\cdots i_r=0}^{d-1} q^{(i_1+\cdots i_r)h-i_2-2i_3-\cdots-(r-1)i_r} \\
&\times \beta_n^{(h,r)}(\frac{w+\alpha_1 i_1+\cdots+\alpha_r i_r}{d}, q^d|\alpha_1,\cdots,\alpha_r).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\beta_n^{(h,r)}(wd,q|\alpha_1,\alpha_2,\cdots,\alpha_r) = [d]^{n-r} &\sum_{i_1,i_2,\cdots i_r=0}^{d-1} q^{(i_1+\cdots i_r)h-i_2-2i_3-\cdots-(r-1)i_r} \\
&\times \beta_n^{(h,r)}(w+\frac{\alpha_1 i_1+\cdots+\alpha_r i_r}{d}, q^d|\alpha_1,\cdots,\alpha_r).
\end{aligned}$$

Remark 3. Note that

$$\lim_{q\to 1}\beta_n^{(h,r)}(w,q|\alpha_1,\alpha_2,\cdots,\alpha_r) = B_n^{(r)}(w|\alpha_1,\alpha_2,\cdots,\alpha_r), \text{ (see Eq. (1)) }.$$

Hence, $\beta_n^{(h,r)}(w,q|\alpha_1,\alpha_2,\cdots,\alpha_r)$ can be considered by the q-analogue of Barnes' multiple Bernoulli polynomials. Now, we will give the inverse formula of Eq. (4).

Indeed we see

$$\sum_{i=0}^{n}\binom{n}{i}(q-1)^i\beta_i^{(h,r)}(q|\alpha_1,\alpha_2,\cdots,\alpha_r)$$
$$=\sum_{i=0}^{n}\binom{n}{i}(q-1)^i\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}[\alpha_1x_1+\cdots+\alpha_rx_r]^i q^{\sum_{j=1}^{r}(h-j)x_j}d\mu_q(x_1)\cdots d\mu_q(x_r)$$
$$=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}q^{n(\alpha_1x_1+\cdots+\alpha_rx_r)}q^{\sum_{j=1}^{r}(h-j)x_j}d\mu_q(x_1)\cdots d\mu_q(x_r)$$
$$=\frac{(n\alpha_1+h)(n\alpha_2+h-1)\cdots(n\alpha_r+h-r+1)}{[n\alpha_1+h][n\alpha_2+h-1]\cdots[n\alpha_r+h-r+1]}.$$

Therefore we obtain the following:

Theorem 3. *For $h\in\mathbb{Z}_{\geq 0}$, we have*

$$\sum_{i=0}^{m}\binom{m}{i}(q-1)^i\beta_i^{(h,r)}(q|\alpha_1,\alpha_2,\cdots,\alpha_r)=\prod_{j=1}^{r}\left(\frac{m\alpha_j+h-j+1}{[m\alpha_j+h-j+1]}\right).$$

Let χ be a Dirichlet character with conductor $d\in\mathbb{Z}_{\geq 0}$. Then we define the generalized Changhee q-Bernoulli numbers as follows: For $m\geq 0$,

$$\beta_{m,\chi}^{(h,r)}(q|\alpha_1,\alpha_2,\cdots,\alpha_r)$$
$$=\underbrace{\int_X\cdots\int_X}_{r\text{ times}}[\alpha_1x_1+\cdots+\alpha_rx_r]^m q^{\sum_{j=1}^{r}(h-j)x_j}\left(\prod_{j=1}^{r}\chi(x_j)\right)d\mu_q(x_1)\cdots d\mu_q(x_r). \tag{6}$$

By simple calculation, we see that

$$\underbrace{\int_X\cdots\int_X}_{r\text{ times}}[\alpha_1x_1+\cdots+\alpha_rx_r]^m q^{\sum_{j=1}^{r}(h-j)x_j}\left(\prod_{j=1}^{r}\chi(x_j)\right)d\mu_q(x_1)\cdots d\mu_q(x_r)$$
$$=[d]^{m-r}\sum_{i_1,\cdots,i_r=0}^{d-1}q^{hi_1+\cdots+(h-r+1)i_r}\left(\prod_{j=1}^{r}\chi(x_j)\right) \tag{7}$$
$$\times\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}[\frac{\alpha_1i_1+\cdots+\alpha_ri_r}{d}+\alpha_1x_1+\cdots+\alpha_rx_r;q^d]^m$$
$$\times q^{\sum_{j=1}^{r}x_j(h-j)d}d\mu_{q^d}(x_1)\cdots d\mu_{q^d}(x_r).$$

By (2),(7), we have the following :

Theorem 4. *For $h \in \mathbb{Z}_{\geq 0}$, we have*

$$\beta_{m,\chi}^{(h,r)}(q|\alpha_1,\alpha_2,\cdots,\alpha_r) = [d]^{m-r} \sum_{i_1,\cdots,i_r=0}^{d-1} q^{hi_1+\cdots+(h-r+1)i_r} \times \left(\prod_{j=1}^{r} \chi(x_j)\right) \beta_m^{(h,r)}(\frac{\alpha_1 i_1+\cdots+\alpha_r i_r}{d}, q^d|\alpha_1,\alpha_2,\cdots,\alpha_r).$$

Remark. By using our formulae in the case of $q = 1$, we can obtain many new formulas which are related to the multiple Barnes' Bernoulli numbers.

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