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A PROPERTY OF AN ARITHMETIC FUNCTION

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A digital arithmetical function described in [1-3] will be defined and new its properties will be described.

Everywhere here and below we shall use the natural number n of the following form

$$n = \sum_{i=1}^{k} a_i.10^{k-i} \equiv \overline{a_1 a_2 ... a_k} \equiv a_1 a_2 ... a_k,$$

the latest notation is for brevity, where a_i is a natural number and $0 \leq a_i \leq 9$ $(1 \leq i \leq k)$.

First, we define a function noted by φ not in the sense of Euler's totient funcion:

$$\varphi(n) = \begin{cases} 0, & \text{if } n = 0 \\ \sum_{i=1}^{m} a_i, & \text{if } n > 0 \end{cases}$$

Let us define a sequence of functions $\varphi_0, \varphi_1, \varphi_2, ...$, where l is a natural number

$$\varphi_0(n) = n,$$

$$\varphi_{l+1} = \varphi(\varphi_l(n)).$$

For every natural number n will exists natural number l so that

$$\varphi_l(n) = \varphi_{l+1}(n) \in \Delta_0 \equiv \{0, 1, 2, ..., 9\},$$

while

$$\varphi_{l-1}(n) \notin \Delta_0.$$

Let $\mathcal{N} = \{0, 1, 2, ...\}$ be the set of the natural numbers. For every natural number $n \geq 1$ we shall construct a set De_n, the elements of which are sequential natural numbers written from left to right in increasing order. Let a_n and b_n be the smallest and the highest elements of Δ_n, respectively. Number a_n will be defined as the smallest natural number a such that $\varphi_{n-1}(a) \notin \Delta_0$ and $\varphi_n(a) \in \Delta_0$, while number b_n will be defined by $b_n = a_{n+1} - 1$.

Obviously, when we construct sets Δ_n, they will satisfy the equality

$$\bigcup_{i=1}^{\infty} \Delta_i = \mathcal{N}.$$

We can seen directly that

$$\Delta_1 = \{10, 11, ..., 18\} = \{10, 11, ..., 18 \times 1\},$$

$$\Delta_2 = \{19, 20, ..., 198\} = \{18 \times 1 + 1, 20, ..., 18 \times 11\}.$$

Let $d_2 = 1$ and for $n \geq 3$

$$d_n = \underbrace{1\,1\,...\,1}_{2d_{n-1} \text{ times}} .$$

We shall prove by induction that for $n \geq 3$

$$\Delta_n = \{1\underbrace{9\,9\,...\,9}_{2d_{n-1} \text{ times}}, 2\underbrace{0\,0\,...\,0}_{2d_{n-1} \text{ times}}, ..., 1\underbrace{9\,9\,...\,9}_{2d_n - 1 \text{ times}}8\}$$

$$= \{18 \times \underbrace{1\,1\,...\,1}_{2d_{n-1} \text{ times}} + 1, ..., 18 \times \underbrace{1\,1\,...\,1}_{2d_n \text{ times}}\}. \qquad (*)$$

When $n = 3$ we obtain $d_3 = 11$ and it can be seen that:
1. if $a = 199 = 198 + 1$, then

$$\varphi_3(a) = \varphi_3(199) = \varphi_2(19) = \varphi_1(10) \equiv \varphi(10) = 1.$$

2. if

$$b = 1\underbrace{9\,9\,...\,9}_{21 \text{ times}}8$$

then

$$\varphi_3(b) = \varphi_2(198) = \varphi_1(18) = 9,$$

$$\varphi_4(b + 1) = \varphi_4(1\underbrace{9\,9\,...\,9}_{22 \text{ times}}) = \varphi_3(199) = \varphi_3(a) = 1,$$

where a satisfies the condition from 1.
3. for every natural number $x: \ a \leq x \leq b: \varphi(x) \leq 198$ and $\varphi_3(x) \in \Delta_0$.

Let us assume that (*) is valid for some natural number n. Now, for the three above steps of the check we obtain as follows.
1. if

$$a = 1\underbrace{9\,9\,...\,9}_{2d_n \text{ times}}$$

then

$$\varphi(a) = 18 \times d_n + 1 = 1\underbrace{9\,9\,...\,9}_{2d_{n-1} \text{ times}} + 1$$

and hence

$$\varphi_{n+1}(a) = \varphi_{n+1}(1\underbrace{9\,9\,...\,9}_{2d_n \text{ times}}) = \varphi_n(1\underbrace{9\,9\,...\,9}_{2d_{n-1} \text{ times}}) = 1$$

by induction assumption.
2. if

$$b = 1\underbrace{9\,9\,...\,9}_{2d_{n+1} - 1 \text{ times}}8,$$

then

$$\varphi(b) = 18 \times d_{n+1} = 1\underbrace{9\ 9\ ...\ 9}_{2d_n - 1 \text{ times}}8 = 9.$$

by induction assumption.

3. for every natural number x : $a \leq x \leq b$:

$$\varphi(x) \leq 1\underbrace{9\ 9\ ...\ 9}_{2d_{n+1} - 1 \text{ times}}8,$$

i.e.,

$$\varphi_{n+1}(x) \in \Delta_0.$$

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