ON SOME PROBLEMS RELATED TO SMARANDACHE NOTIONS

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1. Problem of Number Theory by L. Seagull, Glendale Community College
Let $n$ be a composite integer $>4$. Prove that in between $n$ and $S(n)$ there exists at least a prime number.

Solution:

T.Yau proved that the Smarandache Function has the following property: $S(n) \leq \frac{n}{2}$ for any composite number $n$, because: if $n = pq$, with $p < q$ and $(p,q) = 1$, then

$$S(n) \max(S(p),S(q)) = S(q) \leq \frac{n}{p} \leq \frac{n}{2}.$$  

Now, using Bertrand-Tchebichev's theorem, we get that in between $\frac{n}{2}$ and $n$ there exists at least a prime number.

2. Proposed Problem by Antony Begay

Let $S(n)$ be the smallest integer number such that $S(n)!$ is divisible by $n$, where $m! = 1.2.3.\ldots.m$ (factoriel of $m$), and $S(1) = 1$ (Smarandache Function). Prove that if $p$ is prime then $S(p) = p$. Calculate $S(42)$.

Solution:

$S(p)$ cannot be less than $p$, because if $S(p) = n < p$ then $n! = 1.2.3.\ldots.n$ is not divisible by $p$ ($p$ being prime). Thus $S(p) \geq p$. But $p! = 1.2.3.\ldots.p$ is divisible by $p$, and is the smallest one with this property. Therefore $S(p) = p$.

$42 = 2.3.7, 7! = 1.2.3.4.5.6.7$ which is divisible by 2, by 3, and by 7. Thus $S(42) \leq 7$. But $S(42)$ can not be less than 7, because for example $6! = 1.2.3.4.5.6$ is not divisible by 7. Hence $S(42) = 7$.

3. Proposed Problem by Leonardo Motta

Let $n$ be a square free integer, and $p$ the largest prime which divides $n$. Show that $S(n) = p$, where $S(n)$ is the Smarandache Function, i.e. the smallest integer such that $S(n)!$ is divisible by $n$. 

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Solution:

Because $n$ is a square free number, there is no prime $q$ such that $q^2$ divides $n$. Thus $n$ is a product of distinct prime numbers, each one to the first power only. For example 105 is square free because $105 = 3 \cdot 5 \cdot 7$, i.e. 105 is a product of distinct prime numbers, each of them to the power 1 only. While 945 is not a square free number because $945 = 3^2 \cdot 5 \cdot 7$, therefore 945 is divisible by $3^2$ (which is 9, i.e. a square). Now, if we compute the Smarandache Function $S(105) = 7$ because $7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$ which is divisible by 3, 5, and 7 in the same time, and 7 is smallest number with this property. But $S(945) = 9$, not 7. Therefore, if $n = a \cdot b \cdots p$, where all $a < b < \cdots < p$ are distinct two by two primes, then $S(n) = \max(a, b, \ldots, p = p$, because the factorial of $p$, the largest prime which divides $n$, includes the factors $a, b, \ldots$ in its development: $p! = 1 \cdots a \cdots b \cdots p$.

4. Proposed Problem by Gilbert Johnson

Let $Sdf(n)$ be the Smarandache Double Factorial Function, i.e. the smallest integer such that $Sdf(n)!!$ is divisible by $n$, where $m!! = 1 \cdot 3 \cdot 5 \cdots m$ if $m$ is odd and $m!! = 2 \cdot 4 \cdot 6 \cdots m$ if $m$ is even. If $n$ is an even square free number and $p$ the largest prime which divides $n$, then $Sdf(n) = 2p$.

Solution:

Because $n$ is even and square free, then $n = 2 \cdot a \cdot b \cdots p$ where all $2 < a < b < \cdots < p$ are distinct primes two by two, occurring to the power 1 only. $Sdf(n)$ cannot be less that $2p$ because if it is $2p - k$, with $1 \leq k < 2p$, then $(2p - k)!!$ would not be divisible by $p$.

$$(2p)!! = 2 \cdot 4 \cdots (2a) \cdots (2b) \cdots (2p)$$

is divisible by $n$ and it is the smallest number with this property.