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ABSTRACT LAMBDA POLYNOMIALS AND THEIR CONVOLUTIONS

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1. GENESIS

Abstract Lambda Polynomial Triangles

In the beginning we have unity, 1, and two different entities a, bx (a, b integers $\neq 1$) generating a triangular algebraic "genealogical tree" in Figure 1. This pattern is called the *Abstract Lambda Polynomial Triangle.*

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & a & & bx & & & \\
 & & a^2 & & abx & & (bx)^2 & & \\
 & a^3 & & a^2bx & & a(bx)^2 & & (bx)^3 & \\
a^4 & & a^3bx & & (abx)^2 & & a(bx)^3 & & (bx)^4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots &
\end{array}
$$

Figure 1. Abstract Lambda Polynomial Triangle.

Its law of formation is obvious: members in the "slanting lines" parallel to the left (right) leg of the triangle, which consists of powers of $a(bx)$ with $a^0 = (bx)^0 = 1$, are successive powers of $bx(a)$, each multiplied by $1, a, a^2, a^3, \ldots (1, bx, (bx)^2, (bx)^3, \ldots)$.

Interchanging a and b in Figure 1 produces a similar pattern called the *Abstract Reciprocal Lambda Polynomial Triangle.* Geometrically, this configuration originates from Figure 1 as a reflection about the imagined central vertical axis consisting of powers of abx, i.e. $(abx)^n$ where $n = 0, 1, 2, 3, \ldots$.

When $x = 1$, Figure 1 becomes [6, Figure 8].

Abstract Lambda Polynomials $L_n(x), \ell_n(x)$.

Definition: Recurrence Relation

Two polynomials $L_n(x), \ell_n(x)$– called, respectively, *the Abstract Lambda Polynomial* and *Abstract Reciprocal Lambda Polynomial* – are defined by their recurrence relations thus:

$$
\begin{aligned}
L_{n+2}(x) &= (a+bx)L_{n+1}(x) - abxL_n(x), \quad L_0(x) = 0,\ L_1(x) = 1, && (1.1)\\
\ell_{n+2}(2) &= (b+ax)\ell_{n+1}(x) - abx\ell_n(x), \quad \ell_0(x) = 0,\ \ell_1(x) = 1, && (1.2)
\end{aligned}
$$

wherein the rules played by a and b are interchanged.

From (1.1) and (1.2), the roots of the characteristic equations

$$\lambda^2 - (a+bx)\lambda + abx = 0, \tag{1.1a}$$

$$\lambda^2 - (b+ax)\lambda + abx = 0, \tag{1.2a}$$

are

$$a, bx, \tag{1.1b}$$

$$b, ax. \tag{1.2b}$$

Write

$$\triangle = (a-bx)^2, \ \sigma = (b-ax)^2. \tag{1.3}$$

Simplest $L_n(x)$

Calculations applied successively in (1.1) quickly create an algebraic rhythm producing

$$\begin{aligned}
L_1(x) &= 1 \\
L_2(x) &= a+bx \\
L_3(x) &= a^2+abx+(bx)^2 \\
L_4(x) &= a^3+a^2bx+a(bx)^2+(bx)^3 \\
L_5(x) &= a^4+a^3bx+a^2(bx)^2+a(bx)^3+(bx)^4 \\
L_6(x) &= a^5+a^4bx+a^3(bx)^2+a^2(bx)^3+a(bx)^4+(bx)^5 \\
L_7(x) &= a^6+a^5bx+a^4(bx)^2+a^3(bx)^3+a^2(bx)^4+a(bx)^5+(bx)^6.
\end{aligned} \tag{1.4}$$

Furthermore, notice that

$$L_n(x) = \sum_{r=0}^{n-1} a^{n-1-r}(bx)^r \tag{1.5}$$

which may be demonstrated by induction with (1.1) by means of a little algebraic manoeuvring.

Similar expressions to those in (1.4) arise for the $l_n(x)$ in conformity with (1.2).

Observe that the $L_n(x)$ are pictorially represented as rows in Figure 1. Likewise for the $\ell_n(x)$.

Mutual Reciprocity

Mutually reciprocal relationships

$$L_n(x) = x^{n-1} l_n\left(\frac{1}{x}\right), \tag{1.6}$$

$$l_n(x) = x^{n-1} L_n\left(\frac{1}{x}\right), \tag{1.7}$$

are not difficult to discover.

Abstract Lambda-Lucas Polynomials $M_n(x), m_n(x)$

Appealing as (1.6) and (1.7) are, it is not however the focus of our interest to pursue the mutual reciprocity relationships. Rather, our attention is centred on the new, but simple, *Abstract Lambda-Lucas Polynomial* $M_n(x)$ – the Lucas analogue of $L_n(x)$ – introduced and defined by the recurrence relation

$$M_{n+2}(x) = (a+bx)M_{n+1}(x) - abxM_n(x), \qquad M_0(x) = 2, M_1(x) = a+bx. \tag{1.8}$$

Diagrammatically (Figure 2), the $M_n(x)$ are succinctly realised by means of the Binet form (2.4) which serves also as a definition equivalent to (1.8).

Looking now at the *Abstract Lambda-Lucas Polynomial Diagram* (the two outermost slanting lines in Figure 2),let us, for visual and typographical convenience, temporarily write

$$c = bx, \; U = abx = ac. \tag{1.9}$$

$$\begin{array}{ccccccccccccc}
 & & & & & & 2 & & & & & & \\
 & & & & & a & & c & & & & & \\
 & & & & a^2 & & U & & c^2 & & & & \\
 & & & a^3 & & aU & & cU & & c^3 & & & \\
 & & a^4 & & a^2U & & U^2 & & c^2U & & c^4 & & \\
 & a^5 & & a^3U & & aU^2 & & cU^2 & & c^3U & & c^5 & \\
a^6 & & a^4U & & a^2U^2 & & U^3 & & c^2U^2 & & c^4U & & c^6
\end{array}$$

Figure 2. Abstract Lambda-Lucas Polynomial Diagram (the two outermost slanting lines) enclosing repetitions of Figure 1.

Analysis of Figure 2

Designate as "units" the powers of $U : U^0 = 1,\ U^1,\ U^2,\ U^3, \ldots$ which lie in every second row on the central vertical axis of symmetry in Figure 1. On the central axis in Figure 2, each successive vertex, a power $n \geq 1$ of U separated by two rows, is the apex of a $U-$ copy of the one succeeding it, and therefore of the original pair in Figure 1. Generally, the elements emanating from vertex U^n will be $\{a^m U^n\}_{m=0}^{\infty}$ and $\{(bx)^m U^n\}_{m=0}^{\infty}$. Overall, there is displayed a pleasing herring-bone or chevron pattern.

Mutual Reciprocity

Interchanging a and b in $M_n(x)$, we introduce $m_n(x)$, the *Abstract Reciprocal Lambda-Lucas Polynomial.* Anticipating the Binet form (2.4), we easily establish the mutual reciprocity

$$\begin{aligned} M_n(x) &= x^n m_n\left(\frac{1}{x}\right), \\ m_n(x) &= x^n M_n\left(\frac{1}{x}\right). \end{aligned} \tag{1.10}$$

Historical

Background concepts associated with Plato [8] and Michelangelo [9] are detailed, with references, in [6]. From these philosophic and artistic beginnings the mathematical content of our paper has been wholly abstracted. The linear representation of $M_n(x)$ by the two outside enclosing slanting lines is, however, reminiscent of the diagram given in [8].

Purpose of this Research

Our object is to investigate the basic properties of, and relationships between, $L_n(x)$ and $M_n(x)$. Among these considerations will be rising and descending diagonal polynomials, and certain differentiation features. Aspects of these polynomials will be extended to the convolution polynomials $L_n^{(k)}(x)$ and $M_n^{(k)}(x)$.

Just as the properties of $L_n(x)$ and $M_n(x)$ may be seen against the background of more general theory in [7], so their convolution features may be viewed against the general theory developed in [3]. However, the specialized artistic derivation of lambda polynomials justifies attention to them in their own right.

2. PROPERTIES OF $L_n(x), M_n(x)$

Generating Functions

$$\sum_{n=1}^{\infty} L_n(x)t^{n-1} = \left[1 - \left\{(a+bx)t - abxt^2\right\}\right]^{-1}. \tag{2.1}$$

$$\begin{aligned}\sum_{n=0}^{\infty} M_n(x)t^n &= (1-at)^{-1} + (1-bxt)^{-1} \\ &= \{2-(a+bx)t\}[1-(a+bx)t+abxt^2]^{-1} \qquad (2.2) \\ &= 1 + \frac{1-abxt^2}{1-(a+bxt)+abxt^2}. \qquad (2.2a)\end{aligned}$$

Binet Forms

$$L_n(x) = \frac{a^n - (bx)^n}{a - bx}. \tag{2.3}$$

$$M_n(x) = a^n + (bx)^n. \tag{2.4}$$

Simson Formulas

$$L_{n+1}(x)L_{n-1}(x) - L_n^2(x) = -(abx)^{n-1}. \tag{2.5}$$

$$M_{n+1}(x)M_{n-1}(x) - M_n^2(x) = \Delta(abx)^{n-1}. \tag{2.6}$$

Closed Forms

$$L_n(x) = \sum_{r=0}^{[\frac{n-1}{2}]} (-1)^r (a+bx)^{n-2r-1}(abx)^r \binom{n-1-r}{r}. \tag{2.7}$$

$$M_n(x) = \sum_{r=0}^{[\frac{n}{2}]} (-1)^r \frac{n}{n-r}\binom{n-r}{r}(a+bx)^{n-2r}(abx)^r. \tag{2.8}$$

Guaranteeing the validity of (2.7) requires induction acting on (1.1). To avoid repetition, our proof will be left as a special case $k=0$ in the proof for the convolutions $L_n^{(k)}(x)$.

Trigonometrical Forms

$$L_n(x) = \prod_{r=1}^{n-1}\left\{a+bx-2\sqrt{abx}\cos\left(\frac{k}{n}\pi\right)\right\} \quad (n \geq 2), \tag{2.9}$$

$$M_n(x) = \prod_{r=1}^{n}\left\{a+bx-2\sqrt{abx}\cos\left(\frac{2k-1}{2n}\pi\right)\right\} \quad (n \geq 2), \tag{2.10}$$

where the factors in the products relate to the zeroes of the polynomials. These formulas are adopted from [1,p.118]. Equivalent expressions formed by combining pairs of bracketed factors appropriately are also provided in [1]. For instance, on evaluation,

$$M_4(x) = \left[(a+bx)^2 - 4abx\cos^2\left(\frac{\pi}{8}\right)\right]\left[(a+bx)^2 - 4abx\cos^2\left(\frac{3\pi}{8}\right)\right] = a^4 + bx)^4.$$

Refer also to [6].

Reciprocal Polynomials

All the preceding results should, where relevant, be linked for $\ell_n(x)$ and $m_n(x)$ to the corresponding outcomes for $L_n(x)$ and $M_n(x)$, in particular, (2.1)–(2.10). Observe that the interchange of a and b has no effect on the right-hand sides of (2.5) and (2.6).

A Note on Partial Derivatives

Where as some of our more elementary results are directly analogous to corresponding results in [7], in the case of derivatives it is quite another story. This is because in [7] there are two variables x and y, as against only one variable x here, in accordance with the relationships $x \leftrightarrow a + bx$, $y \leftrightarrow -2bx$.

One result of mild interest is, however, revealed, from (2.1) and (2.2a), namely,

$$\frac{\partial M}{\partial x} - bt\frac{\partial (tL)}{\partial t} = abt^2(1+bxt)L^2, \tag{2.11}$$

where the symbols L, M denote the infinite summations in (2.1) and (2.2).

Relationships between $L_n(x)$ and $M_n(x)$.

Binet forms (2.3) and (2.4) may be employed to establish the following basic connections.

$$L_n(x)M_n(x) = L_{2n}(x), \tag{2.12}$$

$$M_n(x) - abxL_{n-1}(x) = L_{n+1}(x), \tag{2.13}$$

$$M_n(x) + (a+bx)L_n(x) = 2L_{n+1}(x), \tag{2.14}$$

$$M_n(x) - (a+bx)L_n(x) = -2abxL_{n-1}(x), \tag{2.15}$$

$$M_{n+1}(x) - abxM_{n-1}(x) = \Delta L_n(x), \tag{2.16}$$

$$2M_{n+1}(x) - (a+bx)M_n(x) = \Delta L_n(x), \tag{2.17}$$

$$M_{2n}(x) = M_n^2(x) - 2(abx)^n, \tag{2.18}$$

$$M_{2n}(x) = \Delta L_n^2(x) + 2(abx)^n, \tag{2.19}$$

$$2M_{2n}(x) = \Delta L_n^2(x) + M_n^2(x), \tag{2.20}$$

$$L_m(x)M_n(x) + L_n(x)M_m(x) = 2L_{m+n}(x), \tag{2.21}$$

$$M_m(x)M_n(x) + \Delta L_m(x)L_n(x) = 2M_{m+n}(x), \tag{2.22}$$

$$L_m(x)M_n(x) - L_n(x)M_m(x) = 2(abx)^n L_{m-n}(x), \tag{2.23}$$

$$M_m(x)M_n(x) - \Delta L_m(x)L_n(x) = 2(abx)^n M_{m-n}(x). \tag{2.24}$$

While (2.12)-(2.22) are anlogues of [7, (2.8)-(2.18)], the final two formulas have their counterparts in [5]. *En passant*, it may be recorded that (2.14) follows as a direct consequence of (2.1) and (2.2).

3. RISING AND DESCENDING ABSTRACT LAMBDA POLYNOMIALS

A. Polynomials $L_n(x)$

(i) Rising

Construct, in the usual way (see [7,p.218] for references), the *Rising Abstract Lambda Polynomials* $R_n(x)$ of $L_n(x)$. Reading from (1.4), we have $R_1(x) = 1$, $R_2(x) = a$, $R_3(x) = a^2 + bx$, $R_4(x) = a^3 + abx$, $R_5(x) = a^4 + a^2bx + (bx)^2$, $R_6(x) = a^5 + a^3bx + a(bx)^2, \cdots$.

It is an immediate consequence that

$$\left.\begin{aligned} R_n(x) &= aR_{n-1}(x) && (n \text{ even}) \\ R_n(x) &= aR_{n-1}(x) + (bx)^{\frac{n-1}{2}} && (n \text{ odd}) \end{aligned}\right\}, \tag{3.1}$$

$$\text{whence } R_n(x) = a^2 R_{n-2}(x) + a^\delta (bx)^{\frac{n-1}{2}} \tag{3.2}$$

$$\text{where } \delta = \begin{cases} 1 & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases} \tag{3.3}$$

Evidently,

$$R_n(x) = \sum_{r=0}^{[\frac{n-1}{2}]} a^{n-1-2r}(bx)^r. \tag{3.4}$$

Analyzing the expressions for $R_n(x)$ and the form (2.7) for $L_n(x)$, we assert that

$$n \text{ odd}: \quad R_n(x) = \sum_{r=0}^{[\frac{n-2}{3}]} (-1)^r (a^2+bx)^{\frac{n-1}{2}-2r} (a^2bx)^r \begin{pmatrix} \frac{n-1}{2} & -r \\ r & \end{pmatrix}. \tag{3.5}$$

Generating function (n odd) is

$$R \equiv \sum_{n=1}^{\infty} R_{2n-1}(x)t^{n-1} = \left[1 - \left\{(a^2+bx)t - a^2bxt^2\right\}\right]^{-1}, \tag{3.6}$$

whence

$$x(1-2a^2t)\frac{\partial R}{\partial x} - t(1-a^2t)\frac{\partial R}{\partial t} = 0. \tag{3.7}$$

(ii) **Descending**

Reverting to (1.4) and to the standard generation and notation $D_n(x)$ for the polynomials $L_n(x)$, we determine that

$$D_n(x) = a^{n-1}(1-bx)^{-1}$$

whence

$$\frac{dD_n(x)}{dx} = a^{n-1}b(1-bx)^{-2} = a^{n-1}bD_1^2(x), \tag{3.8}$$

and

$$\frac{D_{n+1}(x)}{D_n(x)} = a. \tag{3.9}$$

Write, for the generating function,

$$D \equiv \sum_{n=1}^{\infty} D_n(x)t^{n-1} = (1-bxt)^{-1} \tag{3.10}$$

so that

$$x\frac{\partial D}{\partial x} = t\frac{\partial D}{\partial t}. \tag{3.11}$$

B. Polynomials $M_n(x)$

(i) **Rising**

Designate the rising diagonal polynomials for $M_n(x)$ by $S_n(x)$, i.e., the *Rising Abstract Lambda-Lucas Polynomials.* Then, from (2.4), the first few terms are $S_0(x) = 2$, $S_1(x) = a$, $S_2(x) = a^2 + bx$, $S_3(x) = a^3 + (bx)^2$, $S_4(x) = a^4 + (bx)^3, \cdots$. One doesn't have to be psychic to deduce that

$$S_n(x) = a^n + (bx)^{n-1} \qquad (n \geq 1). \tag{3.12}$$

Moreover,

$$\begin{aligned} \sum_{n=1}^{\infty} S_n(x)t^{n-1} &= a(1-at)^{-1} + bxt(1-bxt)^{-1} \qquad (n \geq 1) \\ &= \frac{a + (1-a)bxt - abxt^2}{1-(a+bx)t + abxt^2}. \end{aligned} \tag{3.13}$$

(ii) **Descending**

Represent by $E_n(x)$ the *Abstract Descending Lambda-Lucas Polynomials* of $M_n(x)$. Then the first few expressions are $E_0(x) = 2 + bx$, $E_1(x) = a + (bx)^2$, $E_2(x) = a^2 + (bx)^3$, $E_3(x) = a^3 + (bx)^4, \cdots$. By inspection,

$$E_n(x) = a^n + (bx)^{n+1} \qquad (n \geq 1). \tag{3.14}$$

Consequently (cf. (2.4)),

$$\sum_{n=1}^{\infty} E_n(x)t^{n-1} = a(1-at)^{-1} + (bx)^2(1-bxt)^{-1}. \tag{3.15}$$

Comments: Attempts to reveal summation formulas for $S_n(x)$ and $E_n(x)$ analogous to those for $R_n(x)$, n odd, in (3.5) have not met with success. Perhaps it would not be too unreasonable to resolve the impasse by reconciling ourselves to claim

$$(i) \qquad S_n(x) = (\text{R.H.S. of}(2.4)) + (bx)^{n-1} - (bx)^n, \tag{3.16}$$

$$(ii) \qquad E_n(x) = (\text{R.H.S. of}(2.4)) + (bx)^{n+1} - (bx)^n. \tag{3.17}$$

4. FIRST CONVOLUTIONS $L_n^{(1)}(x)$, $M_n^{(1)}(x)$

Generating Function Definitions

The *first convolution polynomials* $L_n^{(1)}(x)$, $M_n^{(1)}(x)$ of $L_n(x)$ and $M_n(x)$ respectively, are determined by the generating functions

$$\begin{aligned}
\sum_{n=1}^{\infty} L_n^{(1)}(x)t^{n-1} &= \left[1 - t(a+bx) + abxt^2\right]^{-2}, && (4.1)\\
\sum_{n=0}^{\infty} M_n^{(1)}(x)t^{n} &= \{2-(a+bx)t\}^2\left[1-(a+bx)t + abxt^2\right]^{-2}, && (4.2)\\
\text{with } L_0^{(1)}(x) &= 0, \quad M_0^{(1)}(x) = 4. && (4.3)
\end{aligned}$$

A. **Polynomials $L_n^{(1)}(x)$.**

From (4.1), it transpires that the simplest polynomials $L_n^{(1)}(x)$ are

$$\begin{aligned}
L_1^{(1)}(x) &= 1\\
L_2^{(1)}(x) &= 2(a+bx)\\
L_3^{(1)}(x) &= 3(a+bx)^2 - 2abx\\
L_4^{(1)}(x) &= 4(a+bx)^3 - 6abx(a+bx) && (4.4)
\end{aligned}$$

$$
\begin{aligned}
L_5^{(1)}(x) &= 5(a+bx)^4 - 12abx(a+bx)^2 + 3(abx)^2 \\
L_6^{(1)}(x) &= 6(a+bx)^5 - 20abx(a+bx)^3 + 12(abx)^2(a+bx) \\
L_n^{(1)}(x) &= 7(a+bx)^6 - 30abx(a+bx)^4 + 30(abx)^2(a+bx)^2 - 4(abx)^3.
\end{aligned}
$$

Something of a pattern is apparently emerging in these data.

Taken together, (2.1) and (4.1) immediately lead to

$$L_n(x) = L_n^{(1)}(x) - (a+bx)L_{n-1}^{(1)}(x) + abxL_{n-2}^{(1)}(x). \tag{4.5}$$

Go back now to (2.1). Differentiate partially w.r.t. t and compare coefficients of y^{n-1}. Then

$$(n-1)L_n(x) = (a+bx)L_{n-1}^{(1)}(x) - 2abxL_{n-2}^{(1)}(x). \tag{4.6}$$

Eliminate $L_n(x)$ from (4.6) and (4.7) to derive the *recurrence relation* ($n \longrightarrow n+1$)

$$nL_{n+1}^{(1)}(x) = (n+1)(a+bx)L_n^{(1)}(x) - (n+2)abxL_{n-1}^{(1)}(x). \tag{4.7}$$

Theorem 1

$$L_n^{(1)}(x) = \sum_{r=0}^{[\frac{n-1}{2}]} (-1)^r \binom{n-r}{1}\binom{n-1-r}{r}(a+bc)^{n-2r-1}(abx)^r. \tag{4.8}$$

Proof. Our result is validated by means of induction along with the recurrence relation (4.7).

Remarks:

(i) Pascal's Formula is necessary in the reduction and simplification of the combinatorial products.

(ii) In the proof, the following combinatorial result is required:

$$(N-r)\binom{N-r-1}{r} + 2(N-r)\binom{N-r-1}{r-1} = N\binom{N-r}{r}, \tag{4.8a}$$

which has previously appeared in the Jacobsthal situation in [2, (2.1a)].

Partially differentiating (4.2) with respect to t leads to a useful connective

$$nM_n(x) = (a+bx)L_n^{(1)}(x) - 4abxL_{n-1}^{(1)}(x) + abx(a+bx)L_{n-2}^{(1)}(x). \tag{4.9}$$

B Polynomials $M_n^{(1)}(x)$.

Simplest polynomials $M_n^{(1)}(x)$ are, from (4.2),

$$\begin{aligned}
M_1^{(1)}(x) &= 4(a+bx) \\
M_2^{(1)}(x) &= 5(a+bx)^2 - 8abx \\
M_3^{(1)}(x) &= 6(a+bx)^3 - 16abx(a+bx) \\
M_4^{(1)}(x) &= 7(a+bx)^4 - 26abx(a+bx)^2 + 12(abx)^2 \\
M_5^{(1)}(x) &= 8(a+bx)^5 - 38abx(a+bx)^3 + 36(abx)^2(a+bx) \\
M_6^{(1)}(x) &= 9(a+bx)^6 - 52abx(a+bx)^4 + 75(abx)^2(a+bx)^2 - 16(abx)^3.
\end{aligned} \tag{4.10}$$

Equations (4.1) and (4.2) in conjunction readily reveal that

$$M_n^{(1)}(x) = 4L_{n+1}^{(1)}(x) - 4(a+bx)L_n^{(1)}(x) + (a+bx)^2 L_{n-1}^{(1)}. \tag{4.11}$$

Multiplying numerator and denominator of (2.2) by $1-(a+bx)t+abxt^2$, and simplifying using (4.1), we discover that

$$\begin{aligned}
M_n(x) = 2L_{n+1}^{(1)}(x) - 3(a+bx)L_n^{(1)}(x) &+ \{(a+bx)^2 + 2abx\}L_{n-1}^{(1)}(x) \\
&- (a+bx)abxL_{n-2}^{(1)}(x).
\end{aligned} \tag{4.12}$$

Eliminate $M_n(x)$ from (4.9) and (4.12) to arrive at the recurrence relation

$$\begin{aligned}
2nL_{n+1}^{(1)}(x) = (3n+1)(a+bx)L_n^{(1)}(x) &- \{n(a+bx)^2 + 2(n+2)abx\}L_{n-1}^{(1)}(x) \\
&+ (n+1)abx(a+bx)L_{n-2}^{(1)}(x).
\end{aligned} \tag{4.13}$$

Coming now to the expansion of the right-hand side of (4.2) and gathering like powers of t, we find, using (4.1), that

$$M_n^{(1)}(x) = \sum_{r=0}^{2}(-1)^r 2^{2-r}\binom{2}{r}(a+bx)^r L_{3-r}^{(1)}(x). \tag{4.14}$$

Ultimately, we can fall back on a summation definition for $M_n^{(1)}(x)$ in line with that for Pell-Lucas convolutions [4, (5.1)] when $k=1$. Accordingly, we have

$$M_n^{(1)}(x) = \sum_{j=0}^{n} M_j(x)M_{n-j}(x), \qquad M_0(x) = 2, \tag{4.15}$$

whence, for example, by (2.4) and (4.10),

$$M_2^{(1)}(x) = 2M_0(x)M_2(x) + M_1^2(x) = 4(a^2 + (bx)^2) + (a + bx)^2 = 5(a + bx)^2 - 8abx,$$

as already shown in (4.10).

5. GENERAL CONVOLUTIONS $L_n^{(k)}(x)$, $M_n^{(k)}(x)$

Generating Function Definitions

Proceeding from the first convolution polynomials $L_n^{(1)}(x)$, $M_n^{(1)}(x)$ to the general (k^{th} *order) convolution polynomials* $L_n^{(k)}(x)$, $M_n^{(k)}(x)$ of $L_n(x)$, $M_n(x)$ respectively, we postulate the following

$$\begin{aligned} \sum_{n=1}^{\infty} L_n^{(k)}(x)t^{n-1} &= [1 - (a + bx)t + a\,bxt^2]^{-(k+1)}, && (5.1) \\ \sum_{n=0}^{\infty} M_n^{(k)}(x)t^n &= \{2 - (a + bx)t\}^{k+1}[1 - (a + bx)t + abxt^2]^{-(k+1)}, && (5.2) \end{aligned}$$

with

$$L_0^{(k)}(x) = 0, \quad M_0^{(k)}(x) = 2^{k+1}. \tag{5.3}$$

A. Polynomials $L_n^{(k)}(x)$

Expanding in (5.1) reveals that

$$\begin{aligned} L_1^{(k)}(x) &= 1 \\ L_2^{(k)}(x) &= \binom{k+1}{1}(a+bx) \\ L_3^{(k)}(x) &= \binom{k+2}{2}(a+bx)^2 - \binom{k+1}{1}abx \\ L_4^{(k)}(x) &= \binom{k+3}{3}(a+bx)^3 - 2\binom{k+2}{2}(a+bx)abx \\ L_5^{(k)}(x) &= \binom{k+4}{4}(a+bx)^4 - 3\binom{k+3}{3}(a+bx)^2abx + \binom{k+2}{2}(abx)^2 \\ L_6^{(k)}(x) &= \binom{k+5}{5}(a+bx)^5 - 4\binom{k+4}{4}(a+bx)^3abx + 3\binom{k+3}{3}(a+bx)(abx)^2. \end{aligned} \tag{5.4}$$

Within the definition (5.1) there is contained the immediate deduction

$$L_n^{(k)}(x) = L_n^{(k+1)}(x) - (a + bx)L_{n-1}^{(k+1)}(x) + abxL_{n-2}^{(k+1)}(x). \tag{5.5}$$

Next, differentiate (5.1) partially *w.r.t.* t and compare coefficients of y^{n-1}. It ensues that

$$(n-1)L_n^{(k)}(x) = (k+1)\{(a+bx)L_{n-1}^{(k+1)}(x) - 2abxL_{n-2}^{(k+1)}(x)\}. \tag{5.6}$$

Eliminate $L_n^{(k)}(x)$ from (5.5) and (5.6). Then $(n \longrightarrow n+1)$ we are lift with the recurrence relation $(k \longrightarrow k-1)$

$$nL_{n+1}^{(k)}(x) = (n+k)(a+bx)L_n^{(k)}(x) - (n+2k)abxL_{n-1}^{(k)}(x). \tag{5.7}$$

Putting $k = 0, 1$ in turn in (5.7) brings us back to (1.1) and (4.8) respectively.

Building on the data (5.4) and mindful of (4.8), we can assert:

Theorem 2

$$L_n^{(k)}(x) = \sum_{r=0}^{[\frac{n-1}{2}]} (-1)^r \binom{n+k-1-r}{k}\binom{n-1-r}{r}(a+bx)^{n-2r-1}(abx)^r. \tag{5.8}$$

Proof: Basically, we follow the procedures delineated in the Proof of Theorem 1, namely, induction with appeal to the recurrence relation (5.7).

Remarks similar to those following the Proof of Theorem 1 are now applicable here also. Included, in particular, are the formulas (see [2], (4.12a), (4.12b))

$$k\{\tbinom{N+k-1-r}{k}\tbinom{N-r-1}{r} + 2\tbinom{N-k-1-r}{k}\tbinom{N-r-1}{r-1}\} = N\tbinom{N+k-1-r}{k-1}\tbinom{N-r}{r}, \tag{5.9a}$$

$$N\left[\tbinom{N+k-1-r}{k}\tbinom{N-r}{r} + \tbinom{N+k-1-r}{k-1}\tbinom{N-r}{r}\right] = N\tbinom{N+k-r}{k}\tbinom{N-r}{r}. \tag{5.9b}$$

When $k = 1$, we return to (4.8a) from (4.9a).

B. Polynomials $M_n^{(k)}(x)$.

Calculation of $M_n^{(k)}(x)$ from (5.2) for even small values of n is no easy task. We have, for example,

$$\begin{aligned} M_1^{(k)}(x) &= 2\binom{k+1}{1}(a+bx) \\ M_2^{(k)}(x) &= 2^{k-1}(a+bx)^2\left[4\binom{k+2}{2} + \binom{k+1}{2} - 2\binom{k+1}{1}\binom{k+1}{1}\right] - 2^{k+1}abx\binom{k+1}{1}. \end{aligned}$$

Thus, $k = 2$ gives us $M_2^{(2)}(x) = 18(a+bx)^2 - 24abx$. Or, we may invoke the analogue of [3, (10.1)], namely,

$$M_2^{(2)}(x) = \sum_{j=0}^{2} M_j(x)M_{2-j}^{(1)}(x). \tag{5.9}$$

Generally (cf. [4, (10.1]),

$$M_n^{(k)}(x) = \sum_{j=0}^{n} M_j(x) M_{n-j}^{(k-1)}(x). \tag{5.10}$$

Expanding (5.2) in collaboration with (5.1) shows us that

$$M_n^{(k)}(x) = \sum_{r=0}^{k+1} (-1)^r 2^{k+1-r} \binom{k+1}{r} (a+bx)^r L_{n+1-r}^{(k)}(x), \tag{5.11}$$

where the expansion is expressed as a summation.

6. LAST RITES

Further Developments. Some other avenues left open for investigation include, e.g.

(i) relationships between $L_n(x)$ and $\ell_n(x)$; and between $M_n(x)$ and $m_n(x)$,

(ii) rising and descending diagonal polynomials for convolutions $L_n^{(1)}(x)$ and $M_n^{(1)}(x)$,

(iii) further properties of (a) $L_n^{(1)}(x)$, $M_n^{(1)}(x)$, (b) $L_n^{(k)}(x)$, $M_n^{(k)}(x)$.

Special Case

When $k = 0$, we return, as previously noted, to our stating points, $L_n(x)$ and $M_n(x)$. In this sense, we are like the person in the *Rubaiyat* of Omar Khayyam who

"Came out by the same door as in I went",

except that in the process some pleasurable experiences have been absorbed.

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