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INTERVALS CONTAINING PRIME NUMBERS

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Abstract

For $x > 0$, let $\pi(x)$ be the number of prime numbers not exceeding x. One shows that, for $x \geq 7$, there exists at least one prime number between x and $x + \pi(x)$, thus obtaining a result that is sharper than the one postulated by Bertrand.

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1. INTRODUCTION

Bertrand [1] checked in 1845 that, for every integer n with $2 \leq n \leq$ 3,000,000, the interval $(n, 2n)$ contains at least one prime number. Chebychev [2] gave in 1852 a first proof of this fact. One mentions in [5] the authors of other proofs, and similar results as well. Among these results, let us recall the following:

Nagura [6] proves in 1952 that, for $x \geq 25$, there exists at least one prime number in the interval $\left[x, \frac{6}{5}x\right)$. Rohrbach and Weis [8] show in 1964 that, for every integer $x \geq 118$, the interval $\left(x, \frac{14}{13}x\right)$ contains at least one prime number. Costa Pereira [3] later gives an elementary proof for the existence of a prime number in the interval $\left[x, \frac{258}{257}x\right)$ for $x \geq 485{,}492$.

For $x > 0$, denote by $\pi(x)$ the number of prime numbers not exceeding x. By making use of non-elementary tools, Rosser and Schoenfeld [9] prove several results concerning $\pi(x)$. These results have been recently improved by P. Dusart. More precisely, he shows in [4] that for every integer x we have

$$\pi(x) \geq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}\right) \text{ for } x \geq 32{,}999, \quad (1)$$

$$\pi(x) \leq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\right) \text{ for } x \geq 355{,}991. \quad (2)$$

These inequalities take on in [7] a more convenient form. One shows that for real numbers x we have

$$\pi(x) > \frac{1}{\log x - 1 - 0.6/\log x} \text{ for } x \geq 32{,}999, \quad (3)$$

$$\pi(x) < \frac{x}{\log x - 1 - 1.51/\log x} \text{ for } x \geq 7. \quad (4)$$

Of course, these inequalities easily lead to proofs of the Bertrand type inequalities, that is,

$$\pi(kx) - \pi(x) \geq 1 \text{ for } x \geq n_0(k), \quad (5)$$

the number $n_0(k)$ being determined when k is fixed.

In what follows, we prove a result which is stronger than the results of type (5).

2. THE MAIN RESULT

Theorem. *For every real number $x > 7$ there exists at least one prime number in the interval $(x, x+\pi(x))$.*

Proof. One shows in [9] that for $x \geq 17$ we have

$$\pi(x) > \frac{x}{\log x}, \quad (6)$$

hence it suffices to show that

$$\pi\left(x+\frac{x}{\log x}\right) - \pi(x) > 0. \quad (7)$$

If view of (3), if $x + \pi(x) \geq 32{,}359$, that is, $x > 30{,}000$, we have

$$\pi\left(x+\frac{x}{\log x}\right) > x \cdot \frac{1+\frac{1}{\log x}}{\log x + \log\left(1+\frac{1}{\log x}\right) - 1 - \frac{0.6}{\log\left(x(1+1/\log x)\right)}}.$$

Since for $y > 0$ we have $\log(1+y) < y$, it follows that

$$\log\left(1+\frac{1}{\log x}\right) - \frac{0.6}{\log x + \log(1+1/\log x)} < \frac{1}{\log x} - \frac{0.6}{\log x + 1/\log x},$$

hence

$$\pi\left(x+\frac{x}{\log x}\right) > \frac{x\left(1+\frac{1}{\log x}\right)}{\log x - 1 + \frac{1}{\log x} - \frac{0.6}{\log x + 1/\log x}}.$$

We have $\frac{1}{\log x} - \frac{0.6}{\log x+1/\log x} < \frac{0.41}{\log x}$ for $x \geq 2168$. It follows that

$$\pi\left(x+\frac{x}{\log x}\right) > \frac{x\left(1+\frac{1}{\log x}\right)}{\log x - 1 + \frac{0.41}{\log x}}. \qquad (8)$$

Now (4) and (8) imply that

$$\begin{aligned}\pi\left(x+\frac{x}{\log x}\right) - \pi(x) &> x \cdot \frac{\log^2 x - 2.92\log x - 1.51}{(\log^2 x - \log x + 0.41)(\log^2 x - \log x - 1.51)} \\ &> \frac{x}{(\log x + 1)^2}\end{aligned}$$

for $x \geq 30{,}000$.

It then follows that for $x \geq 30{,}000$ we have

$$\pi(x+\pi(x)) - \pi(x) > \frac{x}{(\log x + 1)^2}. \qquad (9)$$

Now the checking performed for $x < 30{,}000$ finishes the proof. ■

Remark. One proved in [7] that for all integers $x, y \geq 2$ with $\pi(x) \leq y \leq x$ we have

$$\pi(x+y) \leq \pi(x) + \pi(y).$$

This implies that

$$\pi((x+\pi(x)) \leq \pi(x) + \pi(\pi(x)). \qquad (10)$$

Since $\pi(x) \sim x/\log x$, it follows that $\pi(\pi(x)) \sim x/\log^2 x$ hence by (9) and (10) we get

$$\pi((x+\pi(x)) - \pi(x) \sim \frac{x}{\log^2 x}. \qquad (11)$$

It is fairly easy to show that for each fixed natural number n we have

$$\pi((x+\pi(x)) - \pi(x) = x\sum_{k=0}^{n} \frac{a_k}{\log^{k+2} x} + o\left(\frac{x}{\log^{n+3} x}\right). \qquad (12)$$

From (11) we get $a_0 = 1$. It would be interesting to determine the other coefficients $a_1, a_2, \ldots, a_n$ as well.

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