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A NOTE ON SOLVING AND ANALYSING OF THE FULL CUBIC EQUATION

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Introduction: In the present note we use circulant matrices for solving the equation

$$f(x) \equiv x^3 + px^2 + qx + r = 0, \tag{1}$$

when p, q, r are complex numbers. But since the ring of circulant matrices is isomorphic to the ring of tartaglions TAR [1], this means that a tartaglion approach for solving (1) is used, too. When p, q, r are real numbers we analyse four different cases for the roots of (1), which are only possible, giving a criterion for that, when each one of them holds.

1. HOW TO SOLVE THE EQUATION $x^3 + px^2 + qx + r = 0$?

From the history of Matematics it is well known that the basic approach for solving of (1) is to substitute there

$$x = t - \frac{p}{3}, \tag{2}$$

which yields the cubic equation of the kind $t^3 + Pt + Q = 0$.

The last equation has a root t^*, which is given by so called Cardano's formula:

$$t^* = \sqrt[3]{-\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}} + \sqrt[3]{-\frac{Q}{2} - \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}}.$$

This formula was discovered first in 1515 by Scripione del Ferro (1456-1525), but he kept it in secret. Then the formula was rediscovered in 1535 by Niccolo Tartaglia (1500-1557). But Tartaglia also didn't published his formula. This did in 1545 Gerolamo Cardano (1501-1571) who learned it by Tartaglia.

Here we propose another approach for solving (1), which doesen't use the substitution (2), Thus approach is given below.

Let us represent (1) as

$$det(A - x.E^*) = 0, \tag{3}$$

where A is a circulant matrix

$$A = \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix} \tag{4}$$

with unknown complex parameters a, b, c and

$$E^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

In (3) the denotation $det(A - x.E^*)$ is used for the determinant of the matrix

$$A - x.E^* = \begin{pmatrix} a-x & c & b \\ b & a-x & c \\ c & b & a-x \end{pmatrix}. \tag{6}$$

Using the fact that $A - x.E^*$ is a circulant matrix too, and a well known formula for the determinant of an arbitrary circulant matrix (see [2]), we may rewrite (3) in the form

$$(x - (a+b+c)).(x - (a+\omega b+\omega^2 c)).(x - (a+\omega^2 b+\omega c)) = 0, \tag{7}$$

where ω is the basic primitive root of the binomial equation

$$x^3 = 1,$$

i.e.

$$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i.\frac{\sqrt{3}}{2}. \tag{8}$$

On the other hand (after a simple computation of the left-side of (7)) we may rewrite (7) in the form

$$x^3 - 3ax^2 + (3a^2 - 3bc)x + 3abc - a^3 - b^3 - c^3 = 0. \tag{9}$$

Now (7) and (9) imply that all roots of (9) (we denote them x_1^*, x_2^*, x_3^*) are the following:

$$x_1^* = a+b+c \tag{10}$$

$$x_2^* = a+\omega b+\omega^2 c \tag{11}$$

$$x_3^* = a+\omega^2 b+\omega c \tag{12}$$

Since by assumption (1) and (9) represent one and the same equation, the relations:

$$-3a = p;\ 3a^2 - 3bc = q;\ 3abc - a^3 - b^3 - c^3 = r$$

hold. Hence:

$$a = -\frac{p}{3}; \tag{13}$$

$$bc = \frac{p^2 - 3q}{9}; \tag{14}$$

$$b^3 + c^3 = -\frac{2}{27}p^3 + \frac{1}{3}pq - r. \tag{15}$$

Let us introduce numbers D, E, F, depending on p, q, r, by:

$$D = 4.(p^2 - 3q); \tag{16}$$

$$F = -f(\frac{p}{3}) = -\frac{2}{27}p^3 + \frac{1}{3}pq - r \tag{17}$$

$$E = F^2 - 4(\frac{D}{36})^3. \quad (18)$$

Then (14) and (15) yield:

$$b^3 + c^3 = F; \quad (19)$$

$$b^3c^3 = (\frac{D}{36})^3. \quad (20)$$

But (19) and (20) are Viete's formulae for the quadratic equation

$$y^2 - Fy + (\frac{D}{36})^3 = 0 \quad (21)$$

with roots:

$$y_1 = b^3 = \frac{1}{2}F + \frac{1}{2}\sqrt{E};$$

$$y_2 = c^3 = \frac{1}{2}F - \frac{1}{2}\sqrt{E},$$

because of (18).

Hence:

$$b = \sqrt[3]{\frac{1}{2}F + \frac{1}{2}\sqrt{E}}; \quad (22)$$

$$c = \sqrt[3]{\frac{1}{2}F - \frac{1}{2}\sqrt{E}}; \quad (23)$$

Remark 1: The meaning of $\sqrt[3]{\bullet}$ in (22) and $\sqrt[3]{\bullet}$ in (23) are choosen so that (14) is satisfied.

Now (10)-(12) and (22)-(23) give us the final result.

Theorem 1: Let b is given by (22), c is given by (23) and Remark 1 hold. Then all roots of the full cubic equation (1) are given by formulae:

$$x_1^* = a + b + c = -\frac{p}{3} + \sqrt[3]{\frac{1}{2}F + \frac{1}{2}\sqrt{E}} + \sqrt[3]{\frac{1}{2}F - \frac{1}{2}\sqrt{E}};$$

$$x_2^* = a + \omega b + \omega^2 c = -\frac{p}{3} + \omega\sqrt[3]{\frac{1}{2}F + \frac{1}{2}\sqrt{E}} + \omega^2\sqrt[3]{\frac{1}{2}F - \frac{1}{2}\sqrt{E}};$$

$$x_3^* = a + \omega^2 b + \omega c = -\frac{p}{3} + \omega^2\sqrt[3]{\frac{1}{2}F + \frac{1}{2}\sqrt{E}} + \omega\sqrt[3]{\frac{1}{2}F - \frac{1}{2}\sqrt{E}};$$

where:

$$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i.\frac{\sqrt{3}}{2};$$

$$F = -\frac{2}{27}p^3 + \frac{1}{3}pq - r;$$

$$E = (-\frac{2}{27}p^3 + \frac{1}{3}pq - r)^2 - 4(\frac{p^2 - 3q}{9})^3.$$

Remark 2: The result of Theorem 1 is an invariant under changing places of b and c in (22) and (23).

Up to now the coeficient of (1) were arbitrary complex members. But further we shall suppose that they are real numbers.

2. ANALYSING (1) IN THE CASE OF REAL COEFICIENTS

Let us discuss the situation when the coeficients of (1) are real numbers. Of course then (1) has always at least one root, which is a real number. Therefore for the roots of (1) we have only four possible cases. They are described below:

(a_1) The roots of (1) are different real numbers;

(a_2) Only one root of (1) is a real number (the others two are conjugate complex numbers);

(a_3) All roots of (1) are real numbers, but exactly two of them coincide;

(a_4) All roots of (1) are real numbers and they all coincide.

When (a_3) holds, (1) has a double root, which is a real number.

When (a_4) holds, (1) has a triple root, which is a real number.

The following four theorems (we call them E-criterion) show us when each one these cases holds.

Theorem 2. (a_1) holds if and only if (iff) $E < 0$.

Theorem 3. (a_2) holds iff $E > 0$.

Theorem 4. (a_3) holds iff $E = 0$ and $D \neq 0$.

Theorem 5. (a_4) holds iff $E = 0$ and $D = 0$.

Further, as usual, we denote by $\mathcal{R}$ the real number field.

The assertion of Theorem 5 follows immediately from the next

Lemma (1) has a triple root if $q = \frac{p^2}{3}$ and $r = \frac{p^3}{27}$.

Proof of the Lemma: (1) has a triple root $\alpha \in \mathcal{R}$ iff for all $x \in \mathcal{R}$

$$(x - \alpha)^3 = f(x) = x^3 + px^2 + qx + r.$$

Hence:

$$\alpha = -\frac{p}{3}; \; q = \frac{p^2}{3}; \; r = \frac{p^3}{27}$$

and the Lemma is proved.

Proof of Theorem 4: First we observe that: (a_3) holds iff (1) has a double root and (1) has't a triple root. But: (1) has a double (or a triple) root iff

$$\mathcal{R}^*(f(x), \frac{df(x)}{dx}) = 0, \tag{24}$$

where $\mathcal{R}^*(f(x), \frac{df(x)}{dx})$ denotes the resultant of $f(x) = x^3 + px^2 + qx + r$ and

$$\frac{df(x)}{dx} = 3x^2 + 2px + q. \tag{25}$$

Using the well known Sylvester's formula for $\mathcal{R}^*(f(x), \frac{df(x)}{dx})$ (see [3]):

$$\mathcal{R}^*(f(x), \frac{df(x)}{dx}) = detB, \tag{26}$$

where B is the matrix

$$B = \begin{pmatrix} 1 & p & q & r & 0 \\ 0 & 1 & p & q & r \\ 3 & 2p & q & 0 & 0 \\ 0 & 3 & 2p & q & 0 \\ 0 & 0 & 3 & 2p & q \end{pmatrix} \tag{27}$$

we obtain after a computation

$$\mathcal{R}^*(f(x), \frac{df(x)}{dx}) = -18pqr + 4p^3r - p^2q^2 + 4q^3 + 27r^2 = 27.E \tag{28}$$

Therefore (24) and (28) imply:

(1) has a double or a triple root iff $E = 0$.

The Theorem 4 is proved, since if we assume that $E = 0$ and $D = 0$ then Theorem 5 implies: (1) has a triple root,which is not true.

Further we need the relation

$$E = f(x_1).f(x_2), \tag{29}$$

where x_1 and x_2 are roots of

$$\frac{df(x)}{dx} = 3x^2 + 2px + q = 0. \tag{30}$$

One may check (29) directly.

Proof of Theorem 3: Let $E > 0$. We must prove that then (a_2) holds. For this aim we shall consider the following three cases (which are only possible):

(b_1) $x_1 \notin \mathcal{R}$ and $x_2 \notin \mathcal{R}$;

(b_2) $x_1 \in \mathcal{R}$,$x_2 \in \mathcal{R}$ and $x_1 \neq x_2$;

(b_3) $x_1 \in \mathcal{R}$,$x_2 \in \mathcal{R}$ and $x_1 = x_2$.

Let (b_1) hold. Then $D < 0$. Hence, for all $x \in \mathcal{R}$

$$\frac{df(x)}{dx} > 0. \tag{31}$$

As we noted before, there exists $\gamma_1 \in \mathcal{R}$ such that

$$f(\gamma_1) = 0 \tag{32}$$

Let us assume that there exists $\gamma_2 \in \mathcal{R}$, such that $\gamma_2 \neq \gamma_1$ and

$$f(\gamma_2) = 0. \tag{33}$$

Let $\gamma_2 > \gamma_1$. Then according to Rolle's theorem, there exists $\eta \in \mathcal{R}$, such that $\eta \in (\gamma_2, \gamma_1)$ and η is a root of (30). But this contradict to (31).

Therefore (b_1) implies (a_2).

Let (b_2) hold and let $x_1 < x_2$. Then $D > 0$ and from $E > 0$ and (29) we conclude that only the following two cases are possible:

(c_1) $f(x_1) > 0$ and $f(x_2) > 0$;

(c_2) $f(x_1) < 0$ and $f(x_2) < 0$.

For both cases we consider the intervals:

$$I_1 \equiv (-\infty, x_1); \; I_2 \equiv (x_1, x_2); \; I_3 \equiv (x_2, +\infty).$$

It is easy to see that:

$$(\forall x \in I_1 \cup I_3)(\frac{df(x)}{dx} > 0), \; (\forall x \in I_2)(\frac{df(x)}{dx} < 0). \tag{33'}$$

Let (c_1) hold and let $\gamma_1 \in \mathcal{R}$ be a root of (1) (as we noted before, such root always exists). Then $\gamma_1 \notin I_3$, since $f(x)$ is an increasing function on I_3 and $f(x_2) > 0$. Also $\gamma_1 \notin I_2$, since $f(x)$ is a decreasing function on I_2 and: $f(x_1) > 0, f(x_2) > 0$. Obviously $\gamma_1 \neq x_1$ and $\gamma_1 \neq x_2$, because of $f(x_1) > 0$ and $f(x_2) > 0$. Therefore $\gamma_1 \in I_1$.

Let us assume that there exists $\gamma_2 \in \mathcal{R}$, such that $\gamma_2 \neq \gamma_1$ and γ_2 is a root of (1). Then $\gamma_2 \in I_1$, too. Hence, according to Rolle's theorem, there exists a real root of $\frac{df(x)}{dx}$, which lies between γ_1 and γ_2. Therefore this root lies in I_1. But the last conclusion to (33'). Therefore (c_1) implies (a_2).

Let (c_2) hold. Using the same considerations as in the case (c_1), we conclude that (1) has exactly one root $\gamma \in \mathcal{R}$ and moreover $\gamma \in I_3$. The other roots of (1) are conjugate comlpex numbers. Therefore (c_2) implies (a_2), too. Hence, (b_2) implies (a_2).

Let (b_3) hold. Then $D = 0$. Hence for all $x \in \mathcal{R}$

$$\frac{df(x)}{dx} \geq 0. \tag{34}$$

Hence:

$$f(x) \text{ is a nondecreasing function on } \mathcal{R}. \tag{35}$$

Let us assume that there exists $\gamma_1, \gamma_2 \in \mathcal{R}$, such that $\gamma_1 < \gamma_2$ and the equalities (32) and (33) hold. Then (35) implies

$$f(x) = 0, \tag{36}$$

for all $x \in [\gamma_1, \gamma_2]$. But (36) is impossible, since $f(x)$ is a polynomial. Therefore there exists an unique $\gamma \in \mathcal{R}$, which is a root of (1).

Thus we proved that (b_3) implies (a_2) too.

Now, let (a_2) hold. Then we shall prove that $E > 0$ (the case $E = 0$ means that (1) has a double (or a triple) root, but this contradicts to (a_2)).

Let us assume that $E < 0$. Then (18) yields $D > 0$. Hence, $\frac{df(x)}{dx}$ has two different roots: $x_1 \in \mathcal{R}, x_2 \in \mathcal{R}$.

Let $x_1 < x_2$. Therefore $f(x)$ is an increasing function on I_1 and on I_3 too, but is decreasing function on I_2. Using (29) and the assumption that $E < 0$ we conclude that one and only one of the possibilities:

(d_1) $f(x_1) > 0$ and $f(x_2) < 0$;

(d_2) $f(x_1) < 0$ and $f(x_2) > 0$; holds.

Let (d_1) hold. We have that $f(x)$ is a continuous function on $\mathcal{R}$, i.e. on I_1 and on I_3 too. Also $f(x)$ changes its sign on I_1 and I_3 too, since:

$$\lim_{x\to-\infty} f(x) = -\infty; \ \lim f(x)_{x\to+\infty} = +\infty.$$

Therefore (1) has at least two different roots: $\alpha_1 \in I_1; \alpha_2 \in I_3$ (the case $\alpha_1 = \alpha_2$ is impossible, since $I_1 \cap I_3 = \emptyset$). But the last contradicts to (a_2).

Let (d_2) hold. Then for all $x \in (x_1, x_2)$

$$\frac{df(x)}{dx} < 0.$$

Hence:

$$f(x) \text{ is a decreasing function on } I_2. \tag{37}$$

But we have $f(x_1) < 0$. Then there exists $x_1' \in I_2$ such that

$$f(x_1') < 0, \tag{38}$$

since the function $f(x)$ is continous on $\mathcal{R}$.

Also we have $f(x_2) > 0$, Then there exists $x_2' \in I_2$, such that

$$x_2' > x_1' \tag{39}$$

and

$$f(x_2') > 0 \tag{40}$$

since the function $f(x)$ is continous on $\mathcal{R}$.

From (38) and (40) we obtain

$$f(x_2') > f(x_1'). \tag{41}$$

But (39) and (41) contradict to (37). Therefore our assumption, that $E < 0$ is wrong.

Thus we proved that (a_2) implies $E > 0$.

The Theorem 3 is proved.

Proof of the Theorem 2: Let (a_1) hold.

If we assume that $E > 0$, then Theorem 3 implies (a_2). But the last contradicts to (a_1).

If we assume that $E = 0$, then (a_3) or (a_4) hold, according to Theorem 4, or Theorem 5. But each one of the cases: (a_3), (a_4), contradicts to (a_1). Hence: $E < 0$.

Now, let $E < 0$. We must prove that (a_1) holds.

But according to Theorems 3, 4 and 5, each one of the cases: (a_2)-(a_4), is impossible. Then only the case (a_1) remains to be valid. The Theorem 2 is proved.

As a corollary of Theorem 5 we obtain the following

F-criterion: If (a_4) holds, then $F = 0$.

Another corollary (from (18) and Theorem 3) is the following

D-criterion: If $D < 0$, then (a_2) holds.

Thus our investigation is completly finished.

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