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ON A WAY FOR INTRODUCING METRICS IN CARTESIAN PRODUCT OF METRIC SPACES

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1. Introduction

1. Used notations: $\mathcal{R}^n$ – the standard n-dimensional vector space; $\hat{O}$ – the zero vector in $\mathcal{R}^n$; $\hat{+}$ - vector addition in $\mathcal{R}^n$; $\mathcal{R}^n_+$ – the set of all vectors in $\mathcal{R}^n$ with nonnegative components; $\mathcal{P}^n_k$ – the set of all vectors in $\mathcal{R}^n_+$, such that their components are between the numbers: $0, 1, ..., k-1$ ($k \geq 2$ is a fixed integer); $x \succ y$ – relation showing that vectors $x = (x_1, x_2, ..., x_n) \in \mathcal{R}^n_+$ and $y = (y_1, y_2, ..., y_n) \in \mathcal{R}^n_+$ satisfy the condition $x_i \geq y_i$ for each $i = 1, 2, ..., n$; $\oplus$ - the so-called exclusive disjunction; $\mathcal{C}$ - the complex numbers field; $\times$ - the Cartesian product of sets; $\mathcal{C}^n = \underbrace{\mathcal{R} \times \mathcal{R} \times ... \times \mathcal{R}}_{n times}$.

Let $E_1, E_2, ..., E_n$ be metric spaces with corresponding metrics: $d_1, d_2, ..., d_n$. Let us denote by E the cartesian product of sets $E_1, ..., E_n$. Of course, $x \in E$ if and only if (shortly: iff) $x = (x_1, x_2, ..., x_n)$, where $x_i \in E_i, (i = 1, 2, ..., n)$.

Let $\varphi : \mathcal{R}^n_+ \to [0, +\infty)$ be arbitrary function having the following three properties:

$$\varphi(x) = 0 \ \text{ iff } x = \hat{O}, \tag{1}$$

$$\text{for all } x, y \in \mathcal{R}^n_+ : \ \varphi(x \hat{+} y) \leq \varphi(x) + \varphi(y), \tag{2}$$

$$\varphi(x) \geq \varphi(y), \ \text{if } x \succ y. \tag{3}$$

2. Main Theorem

Theorem 1: Let $* : E \times E \to \mathcal{R}^n_+$ be an operation introduced by

$$x * y = (d_1(x_1, y_1), d_2(x_2, y_2), ..., d_n(x_n, y_n)), \tag{4}$$

for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in E$. Then the function $d : E \times E \to [0, +\infty)$ is defined by

$$d(x, y) = \varphi(x * y) \tag{5}$$

is a metric on E.

Proof: First, we shall show that for all $x, y \in E$: $d(x, y) \geq 0$. Indeed, let $x, y \in E$ be arbitrary. Then $x * y \succ \hat{O}$.

Hence, (1), (3), and (5) imply $d(x, y) \geq 0$.

Second, we shall show that $d(x, y) = 0$ iff $x = y$.

Let $x, y \in E$ and $x = y$. Then $x_i = y_i$ and (4) yields $x * y = \hat{O}$, since d_i is a metric on $E_i (i = 1, 2, ..., n)$. Hence, $d(x, y) = 0$, because of (1) and (5).

Let for some $x, y \in E$ we have $d(x,y) = 0$. Then $x * y = \hat{O}$, since (1) holds. Hence for each $i = 1,2,...,n$: $d_i(x_i, y_i) = 0$. Hence $x_i = y_i$, since d_i is a metric on E_i. Hence $x = y$.

The property

$$d(x,y) = d(y,x)$$

follows for every $x, y \in E$ from the equalities $d_i(x_i, y_i) = d_i(y_i, x_i)$ for $i = 1,2,...,n$, which are true since d_i is a metric on E_i.

To prove that d is a metric on E it remains only to prove that for every $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, $z = (z_1, z_2, ..., z_n) \in E$ it is fulfilled

$$d(x,y) + d(x,z) \geq d(y,z). \tag{6}$$

Starting with the inequality

$$d_i(x_i, y_i) + d_i(x_i, z_i) \geq d_i(y_i, z_i),$$

that is true, since d_i is a metric on E_i $i = 1,2,...,n$, we obtain

$$(d_1(x_1,y_1) + d_1(x_1,z_1), (d_2(x_2,y_2) + d_2(x_2,z_2), ..., (d_n(x_n,y_n) + d_n(x_n,z_n))$$

$$\succ (d_1(y_1,z_1), d_2(y_2,z_2), ..., d_n(y_n,z_n). \tag{7}$$

Then (3) and (7) imply

$$\varphi((x * y)\hat{+}(x * z) \geq \varphi(y * z).$$

Hence

$$\varphi(x * y) + \varphi(x * z) \geq \varphi(y * z), \tag{8}$$

because of (2).

Now, (8) proves (6), since we have (5). The Theorem is proved.

Below we give some applications of the above Theorem.

3. Metrics on $\mathcal{P}_k^n$

Let

$$W_k \equiv \{0, 1, ..., k-1\},$$

where $k \geq 2$ is a fixed integer.

We introduce the function $\rho : \mathcal{C} \times \mathcal{C} \to [0, +\infty)$ by

$$\rho(n, v) = |u - v|. \tag{9}$$

If we consider ρ as a function $\rho : W_k \times W_k \to [0, +\infty)$, then ρ is a metric on W_k. Putting

$$E_1 = E_2 = ... = E_n = W_k \tag{10}$$

and

$$d_1 = d_2 = ... = d_n = \rho \tag{11}$$

we have

$$E_1 \times E_2 \times ... \times E_n = \mathcal{P}_k^n$$

and

$$x * y = (|x_1 - y_1|, |x_2 - y_2|, ..., |x_n - y_n|)$$

for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathcal{P}_k^n$, because of (4), (10), and (11).

As a corollary of Theorem 1 and (12) we obtain the following

Theorem 2: Let $\varphi : \mathcal{R}_+^n \to [0, +\infty)$ be arbitrary function having the properties (1) - (3). If $d : \mathcal{P}_k^n \times \mathcal{P}_k^n \to [0, +\infty)$ is the function given by

$$d(x, y) = \varphi(|x_1 - y_1|, |x_2 - y_2|, ..., |x_n - y_n|) \tag{13}$$

for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathcal{P}_k^n$, then d is a metric on $\mathcal{P}_k^n$.

Corollary Function $d : \mathcal{P}_2^n \times \mathcal{P}_2^n \to [0, +\infty)$ introduced by

$$d(x, y) = \varphi(x_1 \oplus y_1, x_2 \oplus y_2, ..., x_n \oplus y_n) \tag{14}$$

is a metric on $\mathcal{P}_2^n$.

Proof: The assertion follows from the equality

$$|u - v| = u \oplus v,$$

that is true for all $u, v \in \mathcal{P}_2^n$.

4. Metrics on $\mathcal{R}^n$ and $\mathcal{C}^n$

The following assertion shows how a lot of the possible metrics on $\mathcal{R}^n$ and $\mathcal{C}^n$ look like.

Theorem 3: Let $\varphi : \mathcal{R}_+^n \to [0, +\infty)$ be arbitrary function having the properties (1) - (3). If $d : \mathcal{R}^n \times \mathcal{R}^n \to [0, +\infty)$ is the function given by (13), then d is a metric on $\mathcal{R}^n$. If $d : \mathcal{C}^n \times \mathcal{C}^n \to [0, +\infty)$ is the function given by (13), then d is a metric on $\mathcal{C}^n$.

Proof: When $d : \mathcal{R}^n \times \mathcal{R}^n \to [0, +\infty)$, we put

$$E_1 = E_2 = ... = E_n = \mathcal{R}^1 \equiv \mathcal{R}$$

and

$$d_1 = d_2 = ... = d_n = \rho,$$

where ρ is given by (9) with $u, v \in \mathcal{R}$. Therefore, the assertion of Theorem 3 for $\mathcal{R}^n = E$ follows from Theorem 1, since ρ is a metric on $\mathcal{R}$.

When $d : \mathcal{C}^n \times \mathcal{C}^n \to [0, +\infty)$, we put

$$E_1 = E_2 = ... = E_n = \mathcal{C}^1 \equiv \mathcal{C}$$

and

$$d_1 = d_2 = ... = d_n = \rho,$$

where ρ is given by (9) with $u, v \in \mathcal{C}$. Therefore, the assertion of Theorem 3 for $\mathcal{C}^n = E$ follows from Theorem 1, since ρ is a metric on $\mathcal{C}$.

5. Conclusion

One may use any norm on $\mathcal{R}^n$, having the property (3), as an example for function φ from Theorem 1.

In particular, the following norms on $\mathcal{R}^n$ are suitable:

$$\varphi(x) = \max_{1 \leq i \leq n} |x_i|, \tag{a_1}$$

$$\varphi(x) = (\sum_{i=1}^{n} |x_i|^\alpha)^{\frac{1}{\alpha}}, \tag{a_2}$$

where $\alpha > 0$ is a real parameter,

$$\varphi(x) = \sum_{i=1}^{n} d_i |x_i|, \tag{a_3}$$

where d_i $(i = 1, 2, ..., n)$ are nonnegative real numbers, such that

$$\sum_{i=1}^{n} d_i > 0.$$

The metrics that corresponds to (a_1) and (a_2) (for $\alpha = 1$ and $\alpha = 2$, respectively) and satisfy (5) are given in [1].

If we use (a_3), when $d_i = 1$ $(i = 1, 2, ..., n)$, then (14) represents the well known Hamming's metrics on $\mathcal{P}_2^n$ (see [2]).

There is an obvious way to generalize the Hamming's metric on $\mathcal{P}_2^n$. We must only put

$$\alpha_i = \beta^{n-i} \; (i = 1, 2, ..., n) \tag{15}$$

in (a_3), where $\beta > 0$ is a fixed real parameter. When $\beta = 1$, using (14) and (a_3) we obtain the Hamming's metric again.

Using (14), (15) and (a_3) we obtain also a class of metrices on $\mathcal{P}_2^n$. When $\beta \geq k$ and β is integer, (a_3) represents an expansion of number $\varphi(x)$ in a positional number system with base β.

Finally, we must note that if in (14) another Boolean function is used, instead of $\oplus$, then the right hand-side of (14) does not represent a metric on $\mathcal{P}_2^n$, because the first property of the metrics "$d(x, y) = 0$ iff $x = y$" is violated.

REFERENCES:

[1] Dieudonné, K., Foundations of Modern Analysis, Academic Press, Paris,1960.

[2] Blahut, R., Theory and Practice of Error Control Codes, Addison-Wesley, Reading, 1984.